Participation Costs, Trend Chasing, and Volatility of Stock Prices

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We analyze an overlapping generations model with fixed costs of stock market participation. Participation in the stock market is determined endogenously and covaries positively with preceding innovations in dividends. The equilibrium share price is positively related to market participation of the same period and to information about future dividends. There is "rational trend chasing" in the sense that, although all agents are rational, market participation rises after an increase of the share price and falls after a decrease. Finally, we show that the endogenous fluctuations of market participation lead to increased volatility of the share price.

There is clear empirical evidence that most agents participate only in very few asset markets and that this holds even for agents with substantial liquid wealth [see e.g., Mankiw and Zeldes (1991); further references can be found in Allen and Gale (1994)]. Moreover, it seems that asset market participation fluctuates. For instance, in informal discussions changes of stock prices are often explained by an inflow or outflow of investors. In this article we analyze an overlapping generations model in which participation in the stock market is determined endogenously and fluctuates over time. The model can help to explain stylized facts of stock markets, such as trend chasing and excess volatility.[1]

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[1] Starting with LeRoy and Porter (1981) and Shiller (1981), there is a huge literature on excess volatility of stock prices, that is, on the fact that the flow of new information about future payoff streams and discount rates seems to be insufficient

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Besides, our model predicts a mean reverting stock price and is, therefore, consistent with the so-called predictability of stock returns [see, e.g., Fama and French (1988a,b, 1989), Poterba and Summers (1988), Fama (1991), Kim, Nelson, and Startz (1991), and Mankiw, Romer, and Shapiro (1991) for different views on this question].

How can we explain limited market participation of rational agents? One possible explanation is fixed costs of market participation, and this is what we assume in our model. In order to invest in a particular market an agent has to bear certain participation or entry costs. Otherwise the agent cannot be active in this market. In addition, we assume that, as a by-product, market participation makes an agent better informed. As an illustration consider the case where an investor has to travel to the marketplace. The travel expenses are fixed entry costs, and once the investor is there she will observe the latest signals and thus will be better informed than agents who have not traveled to the marketplace. In a more modern illustration we can substitute the travel costs by the costs of terminals, software, connecting costs, etc. Another significant example of entry costs follows from the fact that "in order to be active in a market, an agent must initially devote resources to learning about the basic features of the market" [Allen and Gale (1994, p. 934)]. Such basic features of the market include, in particular, the relevant institutional details, as for example, how orders are submitted, what the settlement requirements are, what happens in case of default by other traders, etc. Moreover, a reliable broker has to be identified. Although, due to the necessity to simplify, our model does not directly catch these learning (and search) costs, the motivation for the model is also based on this view of *informational* entry costs. In any case, the assumption that there are fixed costs of market participation and that market participants are better informed than nonparticipants seems to be rather realistic.

In our model there are two assets, risky shares and a riskless asset. Fixed costs of market participation are positive only for the stock market and differ between individual agents. All agents live for two periods, have an initial endowment in the first period and consume only in the second period. Therefore all agents will invest into the riskless asset, but only those with sufficiently low participation costs will participate in the stock market. Moreover, the expected gain (in terms of utility) from stock market participation, and consequently participation itself, varies over time. Dividends are assumed to follow a Markov process. As a result, last period's realized dividends contain information about the probability distribution of future dividends and share prices, and thus about the expected gain from stock

to explain observed stock price volatility. In particular, the debate deals with a variety of econometric aspects. However, recent work, such as LeRoy and Parke (1992), which takes care of econometric problems, seems to confirm the original finding of excess volatility. For references see, for example, LeRoy (1989), Gilles and LeRoy (1991), LeRoy and Parke (1992), and LeRoy and Steigerwald (1995).

market participation. We assume that all agents can, without costs, observe past dividends or prices (or both) and extract the relevant information before deciding about stock market participation. Therefore, in equilibrium, stock market participation will fluctuate over time, reflecting preceding dividends and prices.

Essentially our analysis is based on the view that the agents receive some exogenous information that is relevant for their decisions about stock market participation, and that the expected gain from stock market participation conditional on this information fluctuates over time. Examples, among others, are reports on television or radio stations and information contained in newspapers and magazines. For simplicity we take past dividends as the information which determines the participation decisions, but this assumption should be seen only as a convenient way to model the fact that rational agents will always base their decisions about stock market participation on some relevant information. This simplification makes the model tractable but is not decisive for the results.

What are these results? First, we prove that an equilibrium exists, and that in an equilibrium the share price is positively related to market participation (of the same period) and to information about future dividends. Second, we show that in an equilibrium market participation fluctuates and correlates positively with past dividends. From this we derive *trend chasing* in the sense that market participation rises after an increase of the share price and falls after a decrease.[2] In our model trend chasing is perfectly rational. This implies that we cannot conclude that investors are irrational if we observe trend chasing in an asset market.[3] The last result of this study concerns volatility. We show that the fluctuations of stock market participation increase the volatility of stock prices in a well-defined sense. Thus participation costs and the consequent fluctuations of market participation can be part of an explanation of excess volatility.[4]

There is an emerging literature on the impact of fixed participation costs

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[2] De Long et al. (1990) identify trend chasing with "extrapolative expectations about prices" (p. 379). However, in our view extrapolative price expectations may cause trend chasing but are not identical with it.

[3] Wong (1995) provides empirical evidence of trend chasing in the Hong Kong stock market. For evidence of trend chasing with respect to mutual funds and for its interpretation as a clear sign of irrationality see Patel, Zeckhauser and Hendricks (1991). In De Long et al. (1990) trend chasing is rational for some agents in some periods because rational agents exploit the anticipated *nonrational* trend chasing behavior of "positive feedback traders." Without the nonrational feedback traders trend chasing would not be rational for any agent in their model. In a recent article Brennan and Cao (1996) provide a rigorous model where trend chasing is rational. Although their model is very different from our model, in both models trend chasing is due to informational reasons.

[4] For various approaches to a theoretical explanation of excess volatility see Spiegel (1998) and the references therein. Among these only the explanation of Allen and Gale (1994) (discussed below) is based on costly market participation and is thus related to this article. A model with *exogenous* fluctuations of market participation is contained in Orosel (1997). In contrast, the fluctuations are *endogenous* in the present model.

in asset markets. One line of this literature deals with problems in monetary theory and hence is not closely related to this study [see, e.g., Freeman and Huffman (1991) and Chatterjee and Corbae (1992)]. Another line deals with the effects of market participation on asset prices. Three articles in this line, among others, are Merton (1987), Pagano (1989), and Allen and Gale (1994). Merton (1987) introduced a modified capital asset pricing model (CAPM) where each investor can participate only in markets contained in an exogenous, investor-specific subset of all asset markets. He examines how this modification affects the standard CAPM and shows that limited market participation can explain empirical "anomalies." In contrast to Merton's analysis, which is primarily concerned with the cross-sectional equilibrium structure of asset returns, our model deals with the equilibrium price of one risky asset over time. Furthermore, in our model market participation is determined endogenously. Pagano (1989) analyzes an economy with multiple deterministic equilibria, each being characterized by a constant number of market participants, whereas we will examine the stochastic equilibrium of an economy, where in each period the number of market participants is the realization of an endogenous random variable. Allen and Gale (1994) consider a model with two types of investors and fixed participation costs for the asset market. They derive interesting results on asset price volatility and multiple equilibria. However, as in Pagano (1989), market participation, although different across equilibria, is fixed within each equilibrium. Thus, in contrast to our model, volatility is not related to endogenous fluctuations of market participation. In our equilibrium the added volatility derives from changes in market participation due to innovations in preceding dividends, whereas in Pagano (1989) and in Allen and Gale (1994) it is the level of market participation which influences volatility. Still, our model has two important features in common with Allen and Gale (1994): (i) the view that there are fixed costs of market participation, and (ii) the use of a general utility function rather than a specific one (such as constant absolute risk aversion).

The article is organized as follows. In Section 1 we present the model, and in Section 2 we analyze some properties of an equilibrium and prove its existence. Section 3 deals with rational trend chasing, and Section 4 with volatility. Section 5 contains concluding remarks. Most mathematical proofs are collected in the Appendix.

1. The Model

Time is measured in discrete periods $t = 1, 2, \dots$. There are overlapping generations. All agents have rational expectations, are risk averse, and maximize expected utility. There is one consumption good, which is taken as numeraire in each period. In addition, there are two assets: risky shares and a riskless asset. As in Lucas (1978), shares represent an aggregate firm which has infinite life and produces in each period a stochastic output of the consumption good. This output, which is entirely exogenous, is distributed as dividends at the end of each period. Participation in the stock market requires effort. On the other hand, due to risk aversion agents receive a consumer surplus from buying shares. Thus an agent will participate in the stock market if, in utility terms, the expected consumer surplus exceeds the required effort.

We make several simplifying assumptions to keep the model tractable. However, in Section 5 at the end of this article we will argue that our results can be expected to hold under much more general conditions.

Assumption 1. *There is a riskless asset with a constant (gross) return* $R = 1 + r > 1$. *The supply of this asset is infinitely elastic.*

In order to simplify we impose no short sales restriction, that is, negative investments are feasible. However, for sufficiently large initial endowments no negative investments will occur in equilibrium.

Assumption 2. *The economy is endowed with* $S > 0$ *shares. Each share pays at the end of each period* $t = 1, 2, \ldots$ *random dividends* $\tilde{D}_t$ *with realizations* $d_L \geq 0$ *and* $d_H > d_L$. *Dividends* $\tilde{D}_t$ *follow a first-order Markov process with transition probabilities given by the matrix* $\Pi = \begin{bmatrix} \pi & 1-\pi \\ 1-\pi & \pi \end{bmatrix}$ *where* $\pi \in (\frac{1}{2}, 1)$.

The dividend process is as simple as possible. The justification is to keep the model manageable. Only the assumption $\pi \in (\frac{1}{2}, 1)$ has an economic reason: it formalizes the view that there is some, but not absolute, persistence in the dividend process.

Assumption 3. *In each period* $t = 1, 2, \ldots$ *a new generation, consisting of a continuum of agents which has Lebesgue measure 1 and is uniformly distributed on the interval* $A = [0, 1]$, *enters the economy. Each agent* $a \in A$ *lives for two periods, has an initial endowment of* $w > 0$ *units of the consumption good in the first period of life, and consumes only in the second period. Each agent maximizes expected utility* $V_a = E[U(\tilde{c}_a)] - e_a$, *where* $\tilde{c}_a$ *is (stochastic) consumption in the second period of life,* e_a *is effort expended for participation in the stock market, and* $U : \mathbb{R}_+ \to \mathbb{R}$ *is a von Neumann–Morgenstern utility function with the following properties: (i) for all* $c > 0$, $U(c)$ *is three times differentiable,* $U'(c) > 0$, $U''(c) < 0$, *and* $\alpha'(c) \leq 0$, *where* $\alpha(c) := -\frac{U''(c)}{U'(c)}$ *is absolute risk aversion; (ii)* $\lim_{c \to 0} U'(c) = \infty$.

The overlapping generations assumption allows the combination of finite individual horizons with an infinitely lived economy. Assuming that agents consume only in the second period of their life implies that we abstract from saving decisions and concentrate on portfolio decisions. This is justified because portfolio decisions are much more influential for asset prices. Apart from the plausible assumptions that agents are risk averse ($U''(c) < 0$) and that absolute risk aversion is nonincreasing ($\alpha'(c) \leq 0$) we make no serious restrictions on the utility function. In particular, we do not assume that the utility function is of a special type, such as exhibiting constant absolute or constant relative risk aversion.

Assumption 4. *For agent $a \in A$, participation in the stock market requires effort $k(a) \geq 0$. The function $k(a)$ has the following properties: (i) $k(a) \geq 0$ for all $a \in A$; (ii) there exists an $\varepsilon \in (0, 1)$ such that $k(a) = 0$ for all $a \in [0, \varepsilon]$; (iii) for all $a \in [\varepsilon, 1)$, $k(a)$ is continuous and strictly increasing; (iv) $\lim_{a \to 1} k(a) = \infty$. Participation in the market for the riskless asset requires no effort.*

Agents $a \in A$ who have expended effort $k(a)$ can participate in the stock market and are called *stock market participants* or, for short, *market participants*. Similarly, we will speak of *stock market participation* or, for short, of *market participation*. The effort required for market participation is fixed in the sense that it does not depend on the subsequent investments actually undertaken. We assume that the required effort differs across agents, for example, due to differences in location, interests, capabilities, or education,[5] and we order the agents according to this effort. Assuming that the required effort is zero for a (small) positive fraction ε of each generation simplifies the analysis technically because this assumption implies immediately that market participation will always be strictly positive.[6]

Assumption 5. *At the beginning of period t, $t = 1, 2, \ldots$, all agents observe without costs realized past dividends D_{t-1}. Stock market participants observe, in addition, realized dividends D_t after they have expended effort $k(a)$, but before the stock market opens.*

The current share price can only be observed by market participants[7] (otherwise nonparticipants could infer all the information of market participants). This assumption reflects our view that market participants are better informed than nonparticipants and that nonparticipants receive the

[5] For instance, Mankiw and Zeldes (1991) present empirical evidence (based on 1984 survey data for the United States) that "more highly educated household heads are more likely to be stockholders" (p. 100), and explain this evidence with the argument that "the fixed cost [of participating in the stock market] is lower for the more educated because information acquisition and processing are less costly" (p. 101). Interestingly, the data presented in Mankiw and Zeldes (1991) show for all four income quartiles considered that having an advanced degree reduces the probability of being a stockholder relative to having a college degree only.

[6] Let n_t denote the fraction of market participants of generation t. Then each participating agent has to hold S/n_t shares in equilibrium. Without Assumption 4(ii) a more elaborate technical argument would be needed in the existence proof of the equilibrium in order to ensure that n_t is different from zero.

[7] It would be sufficient to assume that only some market participants observe realized dividends D_t, whereas the other participants rationally infer it from the equilibrium share price.

information of market participants, if at all, only with a time lag. Although knowing dividends D_t eliminates the "dividend risk", market participants still have to bear the "price risk." When they sell their shares after one period the price will depend on the—as yet unknown—realized dividends D_{t+1}.

The decision whether or not to participate in the stock market has to be based on some information, and for simplicity we assume this to be past dividends. Since to a certain degree participation costs can be seen as learning costs [Allen and Gale (1994)], it would be preferable to assume that the participation decision is based on some much more vague information, the evaluation of which requires significantly less knowledge about the stock market. However, then we would have to blow up the (already complicated) model by introducing certain "signals," such as rumors or TV reports, and a specification for extracting information from these signals. In order to avoid these complications and keep the model tractable, we make the simpler assumption that the participation decisions are based on past dividends.

It will turn out that in our model past share prices are informationally equivalent to past dividends, that is, in equilibrium agents can infer past dividends from past share prices and past share prices do not contain any information in addition to past dividends. Therefore it makes no difference whether we assume that agents can observe past dividends, past share prices, or both. We could assume that past prices rather than past dividends are observed, and then prove, in a second step, that this implies that agents can infer past dividends. However, this would complicate the analysis even more without altering the results or adding further insights.

In our model each agent makes sequentially the following decisions and actions: (i) after entering the economy and observing the dividends of the previous period, the agent decides whether or not to participate in the stock market; (ii) if the agent does not participate, she invests all her wealth w into the riskless asset and consumes Rw after one period; (iii) if the agent decides to participate, she expends the required effort, observes the signal, optimally invests her wealth w into a portfolio consisting of shares and the riskless asset, and after one period she liquidates this portfolio and consumes the receipts.

2. Equilibrium

In each period $t = 1, 2, \ldots$ there is a market for the consumption good and for the two assets, and all agents will participate in at least two of these three markets, that is, in the market for the consumption good and in the market for the riskless asset. Because of Assumption 4 it holds that (i) all agents in $[0, \varepsilon] \subset A$ will participate in the stock market in every period; (ii) if in any period it is rational for any given agent $a \in A$ to participate, all agents in $[0, a]$ will participate in this period; and (iii) if in any period it is rational for any given agent $a \in A$ not to participate, no agent in $[a, 1]$ will participate in this period. Thus stock market participation (for short: market participation) in period t can be measured by a positive number $n_t \in A$, implying that all agents in $[0, n_t)$ participate and all agents in $(n_t, 1]$ do not participate. Agent $a = n_t$ is indifferent between participating and not participating, but since the Lebesgue measure of a single agent is zero, her decision is irrelevant for the equilibrium. Because of $A = [0, 1]$, n_t is the fraction of agents who participate in the stock market.

As the consumption good is the numeraire and the rate of return of the riskless asset is exogenous and constant, there is only one price to determine, the price of shares. In any period t, this price will depend (a) on realized dividends D_t, which by assumption can be observed by all stock market participants (for short: market participants); and (b) on market participation n_t, which in turn depends on last period's realized dividends D_{t-1}. If we exclude bubbles and sunspots,[8] the market clearing share price will depend only on D_{t-1} and D_t.

Let $\mathcal{D} := \{d_L, d_H\}$ denote the set of realizations of dividends $\tilde{D}_t$. We restrict our attention to equilibria with the following structure: (i) the market clearing share price of period t is a function of market participation $n_t \in (0, 1]$ and realized dividends $D_t \in \mathcal{D}$, that is, $P_t = P^*(n_t, D_t)$; and (ii) market participation n_t is a function of past dividends D_{t-1}, that is, $n_t = n^*(D_{t-1})$. Substituting $n^*(D_{t-1})$ for n_t in $P^*(\cdot, \cdot)$, we can write P_t as a function of D_{t-1} and D_t, that is, $P_t = p^*(D_{t-1}, D_t)$. This gives the following definition of a (Markov) equilibrium.

Definition 1. *A Markov equilibrium (for short: equilibrium) is a function* $p^*: \mathcal{D}^2 \to \mathbb{R}_+$ *such that for all* $(D_{t-1}, D_t) \in \mathcal{D}^2$ *the stock market clears at* $P_t = p^*(D_{t-1}, D_t)$.

As noted above, all markets clear if the stock market clears. There are two types of equilibrium conditions:

(i) The stock market must clear for each level of market participation [i.e., for $n^*(d_L)$ and $n^*(d_H)$] and for each possible realization of dividends (i.e., for $D_t = d_L$ and $D_t = d_H$).
(ii) Given market clearing share prices, the participation decisions must be utility maximizing.

In order to examine the equilibrium, the first step is to analyze the decisions of market participants. Let $\tilde{P}_t$ denote the random price of shares in period t, $t = 1, 2, \ldots$. The observed price P_t is the realization of $\tilde{P}_t$, and

[8] Sunspot equilibria may exist in our model, although this seems to be improbable [cf. Chiappori and Guesnerie (1991) for a review of the sunspots literature and for references; for a recent article that explains stock price volatility with self-fulfilling beliefs in an overlapping generations model, but does not investigate sunspot equilibria, see Spiegel (1996)]. However, in our model we need no *extransic* uncertainty (i.e., "sun spots") in order to get fluctuations of market participation and share prices. Thus, by Occam's razor, we ignore this possibility.

in an equilibrium p^* we have $\tilde{P}_t = p^*(\tilde{D}_{t-1}, \tilde{D}_t)$. If a market participant buys x shares in period t, her implied stochastic consumption $\tilde{c}_{t+1}$ is

$$\tilde{c}_{t+1} = R(w - P_t x) + (D_t + \tilde{P}_{t+1})x = Rw + (D_t + \tilde{P}_{t+1} - RP_t)x.$$

Thus each market participant maximizes $E\{U[Rw + (D_t + \tilde{P}_{t+1} - RP_t)x] \mid D_t\}$ with respect to x, subject to $\tilde{c}_{t+1} \geq 0$. Because of $\lim_{c\to 0} U'(c) = \infty$, the constraint $\tilde{c}_{t+1} \geq 0$ is not binding.[9] Consequently, the optimal x has to satisfy the first-order condition

$$E\left\{(D_t + \tilde{P}_{t+1} - RP_t)\, U'[Rw + (D_t + \tilde{P}_{t+1} - RP_t)x] \mid D_t\right\} = 0. \quad (1)$$

Since $U'' < 0$, the second order condition is fulfilled. If market participation is $n_t > 0$, the market clearing condition is $x = \frac{S}{n_t}$. Therefore, markets in period t clear if

$$E\left\{(D_t + \tilde{P}_{t+1} - RP_t)\, U'\left[Rw + (D_t + \tilde{P}_{t+1} - RP_t)\frac{S}{n_t}\right] \mid D_t\right\} = 0. \quad (2)$$

In an equilibrium, either $n_t = n^*(d_L) =: m_L$ or $n_t = n^*(d_H) =: m_H$. The equilibrium price $P_t = P^*(n_t, D_t) = p^*(D_{t-1}, D_t)$ can assume the four values $p_{LL} := p^*(d_L, d_L)$, $p_{HL} := p^*(d_H, d_L)$, $p_{LH} := p^*(d_L, d_H)$, and $p_{HH} := p^*(d_H, d_H)$. As a representative example, consider the market clearing condition for the case where previous dividends have been $D_{t-1} = d_H$ and current dividends are $D_t = d_L$. Then, previous dividends being $D_{t-1} = d_H$ implies that current market participation is $n_t = n^*(d_H) = m_H$; and current dividends being $D_t = d_L$ implies (a) that next period's market participation will be $n_{t+1} = n^*(d_L) = m_L$, and (b) that next period's dividends will be $D_{t+1} = d_L$ with probability π and $D_{t+1} = d_H$ with probability $1 - \pi$. Because of (a) and (b), next period's share price will be p_{LL} with probability π and p_{LH} with probability $1 - \pi$. Therefore the return of one share in period t, net of the interest-augmented share price $P_t = p^*(D_{t-1}, D_t) = p^*(d_H, d_L) = p_{HL}$, is $d_L + p_{LL} - Rp_{HL}$ with probability π and $d_L + p_{LH} - Rp_{HL}$ with probability $1 - \pi$. Thus the market clearing condition [Equation (2)] becomes

$$\begin{aligned}\pi(d_L + p_{LL} - Rp_{HL})U'&\left[Rw + (d_L + p_{LL} - Rp_{HL})\frac{S}{m_H}\right]\\ &+ (1-\pi)(d_L + p_{LH} - Rp_{HL})U'\left[Rw + (d_L + p_{LH} - Rp_{HL})\frac{S}{m_H}\right] = 0.\end{aligned}$$

[9] This holds because the optimal x implies $\tilde{c}_{t+1} > 0$. Otherwise (i.e., if $c_{t+1} = 0$ with positive probability) we get $E\{(D_t + \tilde{P}_{t+1} - RP_t)\, U'[Rw + (D_t + \tilde{P}_{t+1} - RP_t)x] \mid D_t\} < 0$ and a decrease of x would increase expected utility.

For each of the four possibilities $(D_{t-1}, D_t) = (d_i, d_j)$, where $i, j = L, H$, we get one market clearing condition like the one for $(D_{t-1}, D_t) = (d_H, d_L)$ derived above. Each market clearing condition follows from Equation (2) and the transition probabilities specified in Assumption 2. A concise way of writing these four market clearing conditions is as follows:

$$\begin{aligned}
&\pi(d_j + p_{jj} - Rp_{ij})U'\left[Rw + (d_j + p_{jj} - Rp_{ij})\frac{S}{m_i}\right] \\
&\quad + (1-\pi)(d_j + p_{jh} - Rp_{ij})U'\left[Rw + (d_j + p_{jh} - Rp_{ij})\frac{S}{m_i}\right] = 0, \\
&\quad i, j, h = L, H, \quad h \neq j, \qquad (3)
\end{aligned}$$

where $Rw + (d_j + p_{jk} - Rp_{ij})\frac{S}{m_i} > 0$ for $i, j, k = L, H$ because $\tilde{c}_{t+1} > 0$ (see note 9). We can rearrange Equation (3) in a way which shows that share prices are equal to the discounted (gross) returns weighted by the associated equivalent martingale probabilities:

$$p_{ij} = \frac{1}{R}\,\frac{(d_j + p_{jj})\ \pi\ U'\left[Rw + (d_j + p_{jj} - Rp_{ij})\frac{S}{m_i}\right] + (d_j + p_{jh})\ (1-\pi)\ U'\left[Rw + (d_j + p_{jh} - Rp_{ij})\frac{S}{m_i}\right]}{\pi\ U'\left[Rw + (d_j + p_{jj} - Rp_{ij})\frac{S}{m_i}\right] + (1-\pi)\ U'\left[Rw + (d_j + p_{jh} - Rp_{ij})\frac{S}{m_i}\right]},$$

$$i, j, h = L, H; \quad h \neq j. \qquad (4)$$

Market participation is endogenous. Therefore, in addition to Equation (3) we have the equilibrium condition that market participation m_i, $i = L, H$, must be the outcome of rational individual decisions. Consequently, given $D_{t-1} = d_i$, agent $a = m_i \geq \varepsilon$ must be indifferent between participating in the stock market and not participating. For period t this implies the equilibrium condition

$$U(Rw) = E\left\{U\left[Rw + (\tilde{D}_t + \tilde{P}_{t+1} - R\tilde{P}_t)\frac{S}{m_i}\right] \mid D_{t-1} = d_i\right\} - k(m_i)$$

$$\text{for } i = L, H.$$

Agents with positive participation costs will participate only if from participating they expect a rent which will compensate them for their participation effort. Because agents are risk averse and absolute risk aversion is nonincreasing, demand for shares is downward sloping.[10] Consequently, market participants do indeed receive a consumer surplus, or rent. Given the participation level m_i, this consumer surplus is $E\{U[Rw + (\tilde{D}_t + \tilde{P}_{t+1} - R\tilde{P}_t)\frac{S}{m_i}] \mid$

[10] Intuitively, this stems from the fact that each additional share increases the risk of the portfolio by more than the previous one. Therefore, the value of the marginal share decreases with the number of shares in the portfolio [for the case of constant absolute risk aversion a more rigorous argument can be found in Orosel (1996, pp. 1397–1398)].

$D_t\} - U(Rw)$. Each agent of generation t has to decide about market participation before observing D_t, and thus before knowing the consumer surplus. However, the agent observes past dividends D_{t-1}. Therefore the agent will derive market participation $n_t = n^*(D_{t-1}) = m_i$ and base the participation decision on the expected consumer surplus conditional on D_{t-1}, which is $E\{U[Rw + (\tilde{D}_t + \tilde{P}_{t+1} - R\tilde{P}_t)\frac{S}{m_i}] \mid D_{t-1}\} - U(Rw)$. Because of Assumption 4 and the equilibrium condition for rational market participation specified above, the expected consumer surplus exceeds the participation costs $k(a)$ for agents $a < m_i$, whereas for agents $a > m_i$ the participation costs exceed the expected consumer surplus. Thus all agents $a < m_i$ will participate, and all agents $a > m_i$ will not participate in the stock market. The resulting participation level is m_i.

Because of Assumption 2 the equilibrium condition for market participation is equivalent to the following two equations, explained below:

$$\begin{aligned} U(Rw) \;=\; & \pi^2\, U\left[Rw + (d_L + p_{LL} - Rp_{LL})\frac{S}{m_L}\right] \\ & + \pi(1-\pi)\, U\left[Rw + (d_L + p_{LH} - Rp_{LL})\frac{S}{m_L}\right] \\ & + \pi(1-\pi)\, U\left[Rw + (d_H + p_{HH} - Rp_{LH})\frac{S}{m_L}\right] \\ & + (1-\pi)^2\, U\left[Rw + (d_H + p_{HL} - Rp_{LH})\frac{S}{m_L}\right] \\ & - k(m_L), \end{aligned} \tag{5}$$

$$\begin{aligned} U(Rw) \;=\; & \pi^2\, U\left[Rw + (d_H + p_{HH} - Rp_{HH})\frac{S}{m_H}\right] \\ & + \pi(1-\pi)\, U\left[Rw + (d_H + p_{HL} - Rp_{HH})\frac{S}{m_H}\right] \\ & + \pi(1-\pi)\, U\left[Rw + (d_L + p_{LL} - Rp_{HL})\frac{S}{m_H}\right] \\ & + (1-\pi)^2\, U\left[Rw + (d_L + p_{LH} - Rp_{HL})\frac{S}{m_H}\right] \\ & - k(m_H). \end{aligned} \tag{6}$$

The derivation of Equations (5) and (6) is based on the transition probabilities (Assumption 2). For instance, if dividends are $D_{t-1} = d_L$ in period $t-1$, dividends in period t will be $D_t = d_L$ with probability π; and thus the share price in period $t+1$ conditional on $D_{t-1} = d_L$ will be $P_{t+1} = p_{LL}$ with probability π^2 and $P_{t+1} = p_{LH}$ with probability $\pi(1-\pi)$. This explains the first two terms on the right-hand side of Equation (5). The other terms in Equations (5) and (6) follow from analogous considerations.

Any solution $(p_{LL},\ p_{HL},\ p_{LH},\ p_{HH},\ m_L,\ m_H)$ of Equations (4)–(6) such that $p_{ij} \in \mathbb{R}_+$ and $m_i \in [\varepsilon, 1]$, $i, j = L, H$ gives an equilibrium with $p^*(d_i, d_j) = p_{ij}$ and $n^*(d_i) = m_i$ for $i, j = L, H$. Before we prove existence, we derive two propositions. For the first one we recall that $r = R - 1$; thus r is equivalent to a riskless rate of interest.

Proposition 1. *In an equilibrium it holds that*

$$p_{ij} \in \left(\frac{d_L}{r}, \frac{d_H}{r}\right) \quad \textit{for } i, j = L, H, \tag{7}$$

$$p_{iH} > p_{iL} \quad \textit{for } i = L, H, \tag{8}$$

$$p_{LH} > p_{HL}. \tag{9}$$

Furthermore, in an equilibrium the price process $\{P_t\}_{t=1}^{\infty}$ *is ergodic; the limiting distribution for the states* $(p_{LL},\ p_{HL},\ p_{LH},\ p_{HH})$ *is given by* $\Pr(p_{LL}) = \Pr(p_{HH}) = \frac{\pi}{2}$ *and* $\Pr(p_{HL}) = \Pr(p_{LH}) = \frac{1-\pi}{2}$.

Proof. See Appendix. ■

If dividends were always low (high) for sure, the share price would be $\frac{d_L}{r}$ ($\frac{d_H}{r}$). This explains Equation (7). Ceteris paribus, higher dividends should imply a higher share price, and Equation (8) shows this to be true. Because of Equation (9), dividends are more influential than market participation for the share price.

Proposition 2. *In an equilibrium,* $p_{Hj} \geq p_{Lj}$ *if and only if* $m_H \geq m_L$*, and* $p_{Hj} > p_{Lj}$ *if and only if* $m_H > m_L$*;* $j = L, H$.[11]

Proof. See Appendix. ■

With higher market participation each participating agent will hold fewer shares in equilibrium. Therefore the marginal share adds less risk to the equilibrium portfolio and consequently the share price will be higher.[12] This explains Proposition 2, which states that ceteris paribus high dividends lead to a high share price in the next period if and only if they induce

[11] The participation levels m_L and m_H are taken as exogenous in the proof. Therefore Proposition 2 holds analogously for market clearing prices associated with nonequilibrium participation levels, that is, for a vector $(p_{LL},\ p_{HL},\ p_{LH},\ p_{HH},\ m_L,\ m_H)$ which fulfills Equation (3) but not Equations (5) and (6).

[12] Level effects could lead to a lower share price. In the proof of Proposition 2 we use the assumption that absolute risk aversion is nonincreasing, and this assumption prevents level effects from generating counterintuitive results.

high market participation (in the next period). Proposition 2 implies that if market participation is constant (i.e., if $m_H = m_L$), share prices depend only on current dividends (i.e., $p_{Hj} = p_{Lj}$, $j = L, H$). Below we show that in an equilibrium market participation is not constant, but fluctuates (Corollary 2) and increases price volatility (Proposition 7). First we prove that an equilibrium exists.

Proposition 3. *There exists an equilibrium.*

Proof. See Appendix. ■

Corollary 1. *For any equilibrium it holds that $m_i > \varepsilon$ for $i = L, H$; that is, some agents with positive participation effort participate in the stock market in every period.*

Proof. The last part of the proof of Proposition 2 implies the corollary. ■

Corollary 1 holds because participating agents receive a consumer surplus. Otherwise they would never expend effort. If agents were risk neutral, consumer surplus would be zero and only agents without participation costs would participate (i.e., $n_t = \varepsilon$ for all $t = 1, 2, \ldots$).

In an equilibrium, share prices are mean reverting; and discounted share prices augmented by dividends will not be a martingale, in general. Thus our model is consistent with the literature on the so-called predictability of stock returns, referred to in the introduction.

3. Equilibrium Participation Levels and Rational Trend Chasing

In our model, trend chasing means that market participation rises after an increase of the share price and falls after a decrease, that is, we can define trend chasing as a positive statistical relationship between the change in market participation $\Delta\tilde{n}_{\tau+1} := \tilde{n}_{\tau+1} - \tilde{n}_\tau$ and the preceding change in the share price $\Delta\tilde{P}_\tau := \tilde{P}_\tau - \tilde{P}_{\tau-1}$, where $\tilde{n}_\tau := n^*(\tilde{D}_{\tau-1})$, $\tau = 1, 2, \ldots$. Frequently, trend chasing is either explained by introducing nonrational traders in addition to rational agents, as in De Long et al. (1990), or is simply regarded as clear evidence of irrational behavior, as in Patel, Zeckhauser, and Hendricks (1991). In still other studies it is interpreted as rational behavior, at least "to a certain extent" [Wong (1995, p. 449)], but no rigorous explanation of why trend chasing can be rational when no non-rational agents exist is given. Recently Brennan and Cao (1996) provided such an explanation. In the context of a noisy rational expectations model with disparately informed agents, they show that in the equilibrium which is not fully revealing "poorly informed agents tend to be 'trend-chasers' " (p. 164). However, when they extend their model to allow for the arrival of private information in more than one period, they have to introduce nonrational "liquidity traders" since otherwise the fully revealing equilibrium, where trend chasing cannot occur, is the only (linear) rational expectations equilibrium. Although their model is very different from our model, their explanation of trend chasing is similar to the one we develop below in the sense that trend chasing is rational and has informational reasons. Empirically, the two explanations, which are not mutually exclusive, could be distinguished by looking at market participation: in contrast to our model, all agents participate in all market sessions in Brennan and Cao (1996).

In this section we show that our model implies trend chasing. Moreover, in our case trend chasing is perfectly rational and there are no nonrational agents in the model. Thus trend chasing does not require nonrational traders and empirical evidence of trend chasing is not, as such, a sufficient proof of irrational behavior.

What is the intuition behind rational trend chasing? We already know that market participation will be higher if expected rents from participating are higher. Since high dividends in period $\tau - 1$ imply a high share price $P_{\tau-1}$ [cf. Equation (8)], we get rational trend chasing if high dividends in period $\tau - 1$ signal high rents from market participation in period τ. Thus trend chasing is generated by the fact that share prices are positively correlated with the rents received by market participants in the next period.

In the following we establish three results which will then imply trend chasing. First, we derive that in any equilibrium it holds that $m_H \neq m_L$, that is, market participation fluctuates. Second, we prove that there always exists an equilibrium such that high dividends induce high market participation in the next period, that is, $m_H > m_L$. Third, we demonstrate that this holds for every equilibrium, if relative risk aversion is not too large. On the basis of these results we will be able to show that trend chasing occurs.

Intuitively, one may expect $m_H > m_L$, that is, that high dividends induce high market participation in the next period. Because of $\pi > \frac{1}{2}$, dividends $\tilde{D}_t$ conditional on $D_{t-1} = d_H$ stochastically dominate dividends $\tilde{D}_t$ conditional on $D_{t-1} = d_L$. But this does not imply $m_H > m_L$. Since dividends are known for sure when trade takes place, they will be completely incorporated in the price of shares and consequently only the old generation $t - 1$ will be affected by the realization of the dividends $\tilde{D}_t$.[13] Thus it needs a deeper analysis to find out whether in an equilibrium it holds that $m_H > m_L$. The first step is Lemma 1. In this lemma we consider the hypothetical situation that market participation is exogenously given and constant, that is, $n_t = M \in (0, 1]$ for all $t = 1, 2, \ldots$, and that share prices are market

[13] To illustrate, assume that at the end of period $t - 1$ unexpectedly we increase D_t by some $\delta > 0$ (with D_τ remaining unaltered for all $\tau \neq t$). Since D_t, respectively $D_t + \delta$, are known when market participants decide about their demand in period t, this will increase the market clearing price to $P_t + \frac{1}{R}\delta$. The old generation will gain from the higher share price, whereas the young generation is indifferent between D_t and P_t on the one hand and $D_t + \delta$ and $P_t + \frac{1}{R}\delta$ on the other.

clearing.[14] The lemma states that in this situation each prospective market participant of period t achieves higher expected utility from consumption (in period $t+1$) conditional on dividends $D_{t-1} = d_H$ than from consumption conditional on dividends $D_{t-1} = d_L$. Because of Assumption 4 this implies that $m_H \neq m_L$ in an equilibrium (Corollary 2 below).

Lemma 1. *Let market participation n_t be given by the constant level $n_t = M \in (0,1]$ for all $t = 1, 2, \ldots$, and let $P_t = p_H$ and $P_t = p_L$ be the market clearing prices associated with $D_t = d_H$ and $D_t = d_L$, respectively. Then,*

$$E[U(\tilde{c}_{t+1}) \mid D_{t-1} = d_H] > E[U(\tilde{c}_{t+1}) \mid D_{t-1} = d_L],$$

where

$$\begin{aligned} &E[U(\tilde{c}_{t+1}) \mid D_{t-1} = d_i] \\ &\quad = \pi \left\{ \pi\, U\left[Rw + (d_i + p_i - Rp_i)\frac{S}{M} \right] \right. \\ &\qquad \left. +\ (1-\pi)\, U\left[Rw + (d_i + p_j - Rp_i)\frac{S}{M} \right] \right\} \\ &\qquad +\ (1-\pi) \left\{ \pi U\left[Rw + (d_j + p_j - Rp_j)\frac{S}{M} \right] \right. \\ &\qquad \left. +\ (1-\pi) U\left[Rw + (d_j + p_i - Rp_j)\frac{S}{M} \right] \right\}, \\ &\quad i, j = L, H, \quad i \neq j. \end{aligned}$$

Proof. See Appendix. ■

Lemma 1 is crucial for our results, but unfortunately it has no straightforward interpretation. In order to gain some intuition we proceed indirectly. First, it is important to see that Lemma 1 compares utilities, not returns. As noted above, agents receive a consumer surplus from buying shares. The expected consumer surplus of an agent of generation t who has observed past dividends D_{t-1}, but not yet current dividends D_t, is $E[U(\tilde{c}_{t+1}) \mid D_{t-1}] - U(Rw) > 0$. Lemma 1 shows that this expected consumer surplus is higher in state $D_{t-1} = d_H$ than in state $D_{t-1} = d_L$. Why should this be so?

From inspection of the terms in brackets in the expression (spelled out in Lemma 1) for $E[U(\tilde{c}_{t+1}) \mid D_{t-1} = d_i]$, $i = L, H$, we see that conditional on D_{t-1} the decision for stock market participation in period t effectively is

[14] For $n_t = M$, $t = 1, 2, \ldots$, the market clearing share price P_t in period t depends on D_t, but not on D_{t-1}, $t = 1, 2, \ldots$. This follows from the fact that Proposition 2 holds analogously for all market clearing prices (associated with any participation levels).

the decision to participate in a compound lottery. This compound lottery depends on the state D_{t-1}. However, the prizes—which are, again, lotteries—are the same in each state D_{t-1}; only the probabilities of receiving these prizes differ. The prizes are S/M units of the two lotteries $\mathcal{L}_H$ and $\mathcal{L}_L$, respectively, defined as follows: One unit of lottery $\mathcal{L}_H$ pays $(d_H + p_H - Rp_H)$ with probability π, and $(d_H + p_L - Rp_H)$ with probability $1-\pi$; one unit of lottery $\mathcal{L}_L$ pays $(d_L + p_L - Rp_L)$ with probability π, and $(d_L + p_H - Rp_L)$ with probability $1 - \pi$. In state $D_{t-1} = d_H$ the compound lottery of stock market participation pays S/M units of lottery $\mathcal{L}_H$ with probability π and S/M units of lottery $\mathcal{L}_L$ with probability $1-\pi$, whereas in state $D_{t-1} = d_L$ these probabilities are reversed, being $1-\pi$ and π, respectively. Since $\pi > \frac{1}{2}$ Lemma 1, that is, $E[U(\tilde{c}_{t+1}) \mid D_{t-1} = d_H] > E[U(\tilde{c}_{t+1}) \mid D_{t-1} = d_L]$, holds if and only if there is a higher consumer surplus associated with lottery $\mathcal{L}_H$ than with lottery $\mathcal{L}_L$. For an interpretation of Lemma 1 we have to understand why this condition is satisfied.

The consumer surplus is due to the fact that in a competitive equilibrium agents pay for each share the reservation price of the *marginal* share, which decreases with the number of shares. Each additional share augments the risk of the portfolio. The higher, at each given number of shares in the portfolio, this risk-augmenting effect of the marginal share is, the faster the reservation price will decrease and thus the higher the consumer surplus will be. Therefore, in order to understand Lemma 1 we have to show that the lottery $\mathcal{L}_H$ has a higher marginal effect on the risk of the portfolio consisting of x units of $\mathcal{L}_H$ than lottery $\mathcal{L}_L$ has on the risk of the portfolio consisting of x units of $\mathcal{L}_L$, where x is any nonnegative real number.

In this nonrigorous, intuitive inspection we ignore several level effects (which complicate the rigorous proof) and normalize the two lotteries $\mathcal{L}_H$ and $\mathcal{L}_L$ to have the same expectation of zero. Subtracting their expected values (which are $\pi p_H + (1-\pi) p_L + d_H - Rp_H$ for $\mathcal{L}_H$, and $(1-\pi) p_H + \pi p_L + d_L - Rp_L$ for $\mathcal{L}_L$, respectively) gives the normalized lotteries $\hat{\mathcal{L}}_H$ and $\hat{\mathcal{L}}_L$: One unit of lottery $\hat{\mathcal{L}}_H$ pays $-\pi(p_H - p_L)$ with probability $1-\pi$, and $(1-\pi)(p_H - p_L)$ with probability π; one unit of lottery $\hat{\mathcal{L}}_L$ pays $-(1-\pi)(p_H - p_L)$ with probability π, and $\pi(p_H - p_L)$ with probability $1-\pi$. The probability distributions of $\hat{\mathcal{L}}_H$ and $\hat{\mathcal{L}}_L$ are depicted in Figure 1. Because of $\pi > 1/2$ the distribution of $\hat{\mathcal{L}}_H$ is necessarily skewed to the left, whereas the distribution of $\hat{\mathcal{L}}_L$ is skewed to the right. Since both lotteries have expected payoffs of zero, they differ only with respect to risk. Unfortunately, the two lotteries cannot be compared according to the Rothschild-Stiglitz criterion of increasing risk [Rothschild and Stiglitz (1970)], which is based on risk aversion only. However, if in addition to risk aversion the rate of absolute risk aversion is nonincreasing (Assumption 3), then marginal utility is convex [i.e., $U''' > 0$; see Equation (23) in the Appendix for a proof] and in this

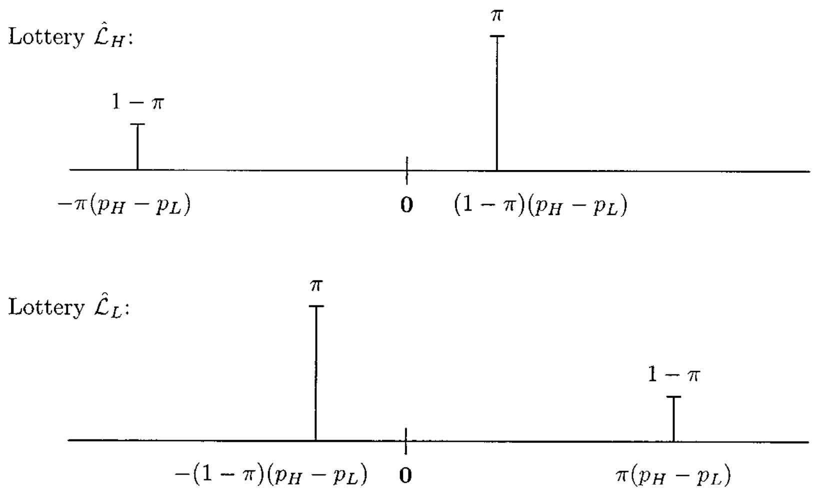

Figure 1
The probability distribution of $\hat{\mathcal{L}}_H$ and $\hat{\mathcal{L}}_L$, respectively
Outcomes are measured on the horizontal axis, probabilities are measured vertically. The positive real numbers p_H and p_L, $p_H > p_L$, denote the share price in the high and low dividend state, respectively. One unit of lottery $\hat{\mathcal{L}}_H$ pays $-\pi(p_H - p_L) < 0$ with probability $1-\pi$, and $(1-\pi)(p_H - p_L) > 0$ with probability π; one unit of lottery $\hat{\mathcal{L}}_L$ pays $-(1-\pi)(p_H - p_L) < 0$ with probability π, and $\pi(p_H - p_L) > 0$ with probability $1-\pi$.

case we can, in fact, compare the risk of the two lotteries. Intuitively, with convex (and decreasing) marginal utility agents will prefer a distribution that is skewed to the right (where marginal utility is less steep) to a distribution that has the same mean and is skewed to the left. Because of this, they will regard lottery $\hat{\mathcal{L}}_H$ as more risky than lottery $\hat{\mathcal{L}}_L$. The following remark shows this intuition to be correct. In this remark we contrast the two portfolios consisting of x units of lottery $\hat{\mathcal{L}}_H$ and lottery $\hat{\mathcal{L}}_L$, respectively, where x is any nonnegative real number.

Remark 1. *Consider two portfolios, one consisting of x units of lottery $\hat{\mathcal{L}}_H$ (called $\hat{\mathcal{L}}_H$ portfolio) and the other consisting of of x units of lottery $\hat{\mathcal{L}}_L$ (called $\hat{\mathcal{L}}_L$ portfolio), where $x > 0$. For each utility function $U(c)$ which is three times differentiable and satisfies $U'(c) > 0$, $U''(c) < 0$, and $U'''(c) > 0$, the following holds:*

(i) For any $x > 0$, the $\hat{\mathcal{L}}_H$ portfolio is more risky than the $\hat{\mathcal{L}}_L$ portfolio.
(ii) At any $x > 0$, a marginal increase in x decreases the expected utility of the $\hat{\mathcal{L}}_H$ portfolio more than the expected utility of the $\hat{\mathcal{L}}_L$ portfolio, that is, the marginal effect of the lottery $\hat{\mathcal{L}}_H$ on the risk of the $\hat{\mathcal{L}}_H$ portfolio is larger than the marginal effect of the lottery $\hat{\mathcal{L}}_L$ on the risk of the $\hat{\mathcal{L}}_L$ portfolio.

Proof. See Appendix. ■

Because of Remark 1 we expect a higher consumer surplus to be associated with lottery $\mathcal{L}_H$ than with lottery $\mathcal{L}_L$. Consequently, at the respective equilibrium price agents will prefer a compound lottery which gives more weight to lottery $\mathcal{L}_H$ than to lottery $\mathcal{L}_L$, and this implies Lemma 1. An immediate consequence of Lemma 1 is the following corollary.

Corollary 2. *For any equilibrium it holds that* $m_H \neq m_L$.

Proof. Indirect. Assume that $m_L = m_H = M$ in an equilibrium. Then $p_{HH} = p_{LH} = p_H$ and $p_{HL} = p_{LL} = p_L$. Using the notation of Lemma 1, Equations (5) and (6) imply $E[U(\tilde{c}_{t+1}) \mid D_{t-1} = d_H] = U(Rw) + k(M) = E[U(\tilde{c}_{t+1}) \mid D_{t-1} = d_L]$. This contradicts Lemma 1 and proves the proposition. ■

In spite of Lemma 1 we cannot prove that for every equilibrium it holds that $m_H > m_L$. However, we can derive two somewhat weaker results: (i) there always exists an equilibrium with $m_H > m_L$; (ii) if relative risk aversion is not too large, $m_H > m_L$ holds for every equilibrium. The following proposition states the first result.

Proposition 4. *There exists an equilibrium* p^* *such that* $m_H > m_L$ *and consequently* $p_{HH} > p_{LH} > p_{HL} > p_{LL}$.

Proof. See Appendix. ■

For the remaining part of our analysis we introduce an additional assumption concerning the utility function $U(c)$. Let $\beta(c) := -\frac{U''(c)}{U'(c)}c$ denote relative risk aversion. We assume

$$\beta(c) < 1 + \frac{Rw}{c - Rw} \qquad \text{for all } c > Rw. \tag{10}$$

For sufficiently large Rw this assumption is not implausible. Since $\frac{d[zU'(Rw+z)]}{dz} = U'(Rw+z) + zU''(Rw+z) = U'(Rw+z)\left[1 - \frac{z}{Rw+z}\beta(Rw+z)\right]$, Equation (10) is equivalent to

$$\frac{d[zU'(Rw+z)]}{dz} = U'(Rw+z) + zU''(Rw+z) > 0 \quad \text{for all } z > -Rw \tag{11}$$

(for $z \leq 0$, i.e., $c \leq Rw$, Equation (11) follows from $U' > 0$ and $U'' < 0$). Because of Equation (1) the inequality of Equation (11) is equivalent to the following statement: If we increase, ceteris paribus, the return of shares in any single state of the world which occurs with positive probability, then demand for shares will increase. Therefore Equation (10) seems to be an acceptable assumption. Although this assumption may not be necessary for the results derived below, it will be of great help in the analysis.[15]

For the following lemma we regard market participation as exogenous and analyze its effects on share prices. We consider two exogenous participation levels, M_L and M_H, where $\varepsilon \leq M_i \leq 1$ for $i = L, H$ and where market participation n_t in period t is exogenously given by $n_t = M_i$ if $D_{t-1} = d_i$, $i = L, H$. Let $P = (P_{LL}, P_{HL}, P_{LH}, P_{HH})$ denote the market clearing share prices associated with the exogenously given participation levels (M_L, M_H), that is, P_{ij} will clear the stock market in period t if $D_{t-1} = d_i$ (implying $n_t = M_i$) and $D_t = d_j$, $i, j = L, H$. The following lemma shows that P is a strictly increasing function of (M_L, M_H), that is, an increase in any of the two (exogenous) participation levels will lead to an increase in all four share prices.

Lemma 2. *If Equation (10) holds, market clearing share prices* $P = (P_{LL}, P_{HL}, P_{LH}, P_{HH})$ *are differentiable, positive functions of the market participation levels* $(M_L, M_H) \in [\varepsilon, 1]^2$ *and*

$$\frac{\partial P_{ij}}{\partial M_k} > 0; \qquad i, j, k = L, H. \tag{12}$$

Proof. See Appendix. ■

With the help of Lemmas 1 and 2 we are now ready to prove that Equation (10), that is, the assumption that relative risk aversion is not too large, is a sufficient condition for $m_H > m_L$. Below we will show that this, in turn, implies trend chasing.

Proposition 5. *If Equation (10) holds, then* $m_H > m_L$ *and consequently* $p_{HH} > p_{LH} > p_{HL} > p_{LL}$ *in every equilibrium.*

Proof. See Appendix. ■

Proposition 5 demonstrates that under plausible assumptions we get $m_H > m_L$. It is easy to show that an equilibrium with $m_H > m_L$ implies (rational) trend chasing. Table 1 lists (i) all eight possibilities for $(P_{t-1}, P_t) = (p_{hi}, p_{ij})$, $h, i, j = L, H$; (ii) the associated vector $(n_t, n_{t+1}) = (n_i, n_j)$, $i, j = L, H$; (iii) the implied pair $(sgn\Delta P_t, sgn\Delta n_{t+1})$; and (iv) the (limiting) probability of each observation. This table immediately implies Proposition 6, which clearly shows that in an equilibrium with $m_H > m_L$ there is rational trend chasing. Intuitively there is a positive correlation between last period's share price and the expected consumer surplus from market

[15] If Equation (10) is assumed, we can replace $\alpha'(c) \leq 0$ in Assumption 3 by the weaker requirement $U'''(c) > 0$ without changing any of the results. This holds because, as does $\alpha'(c) \leq 0$, Equation (11) implies $\partial F_L(y, n)/\partial y < 0$ in the proof of Proposition 2 and $\partial F(y)/\partial y < 0$ in the proof of Claim 1 (which is part of the proof of Lemma 1).

Table 1
Trend chasing

(P_{t-1}, P_t)	(n_t, n_{t+1})	$(sgn\Delta P_t,\ sgn\Delta n_{t+1})$	Probability
(p_{HH}, p_{HH})	(m_H, m_H)	(0,0)	$\frac{\pi^2}{2}$
(p_{HH}, p_{HL})	(m_H, m_L)	(–,–)	$\frac{\pi(1-\pi)}{2}$
(p_{LH}, p_{HH})	(m_H, m_H)	(+,0)	$\frac{\pi(1-\pi)}{2}$
(p_{LH}, p_{HL})	(m_H, m_L)	(–,–)	$\frac{(1-\pi)^2}{2}$
(p_{HL}, p_{LH})	(m_L, m_H)	(+,+)	$\frac{(1-\pi)^2}{2}$
(p_{HL}, p_{LL})	(m_L, m_L)	(–,0)	$\frac{\pi(1-\pi)}{2}$
(p_{LL}, p_{LH})	(m_L, m_H)	(+,+)	$\frac{\pi(1-\pi)}{2}$
(p_{LL}, p_{LL})	(m_L, m_L)	(0,0)	$\frac{\pi^2}{2}$

Table 1 lists in the first column all eight possibilities for the share prices $(P_{t-1}, P_t) = (p_{hi}, p_{ij}), h, i, j = L, H$ in period $t-1$ and t, respectively; in the second column the associated vector $(n_t, n_{t+1}) = (n_i, n_j), i, j = L, H$ of stock market participation in period $t-1$ and t, respectively; in the third column the implied pair $(sgn\Delta P_t, sgn\Delta n_{t+1})$ of the sign of the price change in period t and of the asssociated change in market participation in the following period $t+1$; and in the fourth column the (limiting) probability of each observation.

participation in this period, and this implies a positive correlation between last period's share price and this period's level of market participation, that is, (rational) trend chasing.

Proposition 6. *In an equilibrium with $m_H > m_L$,*

(i) $\Delta n_{t+1} \neq 0$ *implies* $sgn\Delta n_{t+1} = sgn\Delta P_t$,

(ii) $\Delta P_t \geq 0$ *implies* $\Delta n_{t+1} \geq 0$,

(iii) $\Delta P_t = 0$ *implies* $\Delta n_{t+1} = 0$,

(iv) $\Delta P_t \leq 0$ *implies* $\Delta n_{t+1} \leq 0$,

(v) $Pr(sgn\Delta n_{t+1} = sgn\Delta P_t) = 1 - \pi(1-\pi) \in (\frac{3}{4}, 1)$.

Proof. See Table 1. ■

4. Volatility

In principle, fluctuations of market participation may increase or decrease share price volatility. It is not easy to analyze this question in our model for two reasons. First, there is the conceptual difficulty that even with constant market participation volatility of share prices will depend on the chosen constant level of market participation, and that it is not clear which constant level should be taken for the comparison. Second, since we have no explicit solution for the model, it is difficult to derive any rigorous results. In spite of these difficulties we will nevertheless be able to show that the endogenous fluctuations of market participation increase share price volatility in our model. This is relevant for the excess volatility debate because it implies that, due to endogenously changing levels of market participation, the volatility of stock prices can exceed the volatility of the present value of expected future dividends even if all agents are rational and discount rates are constant over time. Thus our analysis can contribute, in part, to an explanation of the observed "excess volatility" of stock prices.

Consider an equilibrium p^* with $p_{LL} < p_{HL} < p_{LH} < p_{HH}$ and associated equilibrium participation levels $m_L < m_H$. We want to show that the equilibrium prices $p := (p_{LL}, p_{HL}, p_{LH}, p_{HH})$ are "more volatile" than the market clearing prices $P = (P_{LL}, P_{HL}, P_{LH}, P_{HH})$ associated with some constant exogenous market participation level $M = M_L = M_H$. Because of $M_L = M_H$, we have $P_{HL} = P_{LL}$ and $P_{LH} = P_{HH}$. For any $M \in [\varepsilon, 1]$ let $P_L(M)$ denote the associated market clearing price $P_{HL} = P_{LL}$ if dividends are low ($D_t = d_L$), and let $P_H(M)$ denote the associated market clearing price $P_{LH} = P_{HH}$ if dividends are high ($D_t = d_H$). Market clearing implies $P_L(M) < P_H(M)$, analogous to Equation (8). The following definition makes precise when we regard the share prices $P(M) := [P_L(M), P_H(M)]$ for a given M as "less volatile" than the equilibrium prices $p = (p_{LL}, p_{HL}, p_{LH}, p_{HH})$.

Definition 2. *Let $p = (p_{LL}, p_{HL}, p_{LH}, p_{HH})$, $p_{LL} < p_{HL} < p_{LH} < p_{HH}$, be equilibrium prices corresponding to equilibrium participation levels (m_L, m_H), and let $P(M) = [P_L(M), P_H(M)]$ be the market clearing prices associated with a given constant participation level $M \in [\varepsilon, 1]$. Then the prices $P(M)$ are less volatile than the equilibrium prices p if the following two conditions hold:*

$$p_{LL} < P_L(M) < P_H(M) < p_{HH}, \tag{13}$$

$$P_H(M) - P_L(M) < \min\,(p_{LH} - p_{LL},\ p_{HH} - p_{HL}). \tag{14}$$

If Equations (13) and (14) hold, then the variances conditional on D_{t-1}, the unconditional variance, and the range of the equilibrium share prices p all exceed the respective volatility measures of $P(M)$.

The following proposition shows that the endogenous fluctuations of market participation levels do, in fact, increase the volatility of equilibrium share prices.

Proposition 7. *Assume Equation (10) and let p^* be an equilibrium with participation levels (m_L, m_H) and prices $(p_{LL}, p_{HL}, p_{LH}, p_{HH})$. Then there exists an interval $\mathcal{I} \subset [m_L, m_H]$ of positive length such that for all participation levels $M \in \mathcal{I}$ the associated market clearing prices $[P_L(M), P_H(M)]$ are less volatile than the equilibrium prices $(p_{LL}, p_{HL}, p_{LH}, p_{HH})$.*

Proof. See Appendix. ∎

5. Concluding Remarks

There is an emerging literature on the effects of fixed costs of participation in asset markets. This literature demonstrates that taking participation costs into account can help to explain stylized facts. Our analysis adds to this literature. For an overlapping generations model in which dividends follow a Markov process we have shown that participation costs generate endogenous fluctuations of participation levels in the stock market. Specifically, participation levels covary positively with preceding innovations in dividends. This leads to rational trend chasing: participation in the stock market rises after an increase in the share price and falls after a decrease, even though all agents are perfectly rational. Moreover, the endogenous fluctuations of stock market participation add to the effects of dividends on share prices and increase volatility. Therefore, fixed participation costs and the associated fluctuations of market participation can be part of an explanation of the observed "excess volatility" of share prices.

How robust are these results? Two assumptions of the model are particularly restrictive: that the risk-free rate of return is constant, and that dividends follow a simple Markov process. Can we expect our results to hold if we replace these assumptions by more realistic ones?

If the supply of the riskless asset is not infinitely elastic at a constant rate of return, an increase in the demand for it will raise its "price," that is, decrease the riskless rate of return. How will this affect our results? The driving force in the model is that innovations in dividends lead to innovations in expected rents from participating in the stock market, with high dividends implying high expected rents and thus high market participation in the next period. If the risk-free rate decreases when demand for the riskless asset increases, the difference in the rents from market participation between states with high and low levels of market participation will be reduced. However, it will not be eliminated or reversed, since ceteris paribus the risk-free rate will only be higher when demand for the riskless asset is lower, which implies that demand for the alternative asset and thus participation in the stock market has to be higher. Consequently, with a variable risk-free rate, high (low) dividends will still imply high (low) market participation, and trend chasing follows. Therefore we can expect our results—which are qualitative, not quantitative results—to hold as well when the risk-free rate is variable.

There are also reasons to be confident that our results generalize to a wider class of dividend processes. The essential feature of the model's dividend process—which is responsible for rational trend chasing—is that high (low) dividends signal high (low) rents from market participation in the next period. Although this will not be true for all dividend processes, we have no reason to believe that this is a very peculiar property only of the dividend process chosen for simplicity in our model. For instance, dividends may follow a martingale where the conditional distribution of innovations in dividends is different for different levels of present dividends. In general, a higher conditional variance of returns (dividends plus capital gains) will be associated with higher risk and thus with a higher rent for investors. If in equilibrium the conditional variance of future returns is increasing in the current dividend level and if dividends and prices are positively correlated, there will be a positive correlation between present prices and tomorrow's rents from market participation. Therefore trend chasing follows. This illustrates that rational trend chasing (and "excess volatility") can be expected to be generated by a wide class of dividend processes. However, a rigorous analysis of a model with a general dividend process would be extremely difficult if not impossible to perform, and without such an analysis we cannot identify theoretically the set of dividend processes that will lead to trend chasing.

Finally, the other assumptions of our model are less heroic, and it is easily seen that our results would survive modifications of these assumptions as well. Thus our results seem to be robust. Participation costs and participation levels can indeed help to explain trend chasing and "excess volatility" and should be taken into account, if possible, in empirical studies.

6. Appendix

Proof of Proposition 1. Define $p^{\max} := \max_{i,j\in\{L,H\}} p_{ij}$ and $p^{\min} := \min_{i,j\in\{L,H\}} p_{ij}$. Then Equation (4) implies $p_{ij} \leq R^{-1}(d_H + p^{\max})$, and thus $p^{\max} \leq \frac{d_H}{r}$. Analogously, $p^{\min} \geq \frac{d_L}{r}$. This gives

$$\frac{d_L}{r} \leq p_{ij} \leq \frac{d_H}{r} \qquad \text{for } i, j = L, H. \tag{15}$$

First we prove Equation (8) by contradiction. For notational convenience let $\kappa(j)$, $j \in \{L, H\}$ be defined by $\kappa(L) = H$ and $\kappa(H) = L$. Assume $p_{LH} \leq p_{LL}$. Then $d_L + p_{LL} - Rp_{LL} \geq d_L + p_{LH} - Rp_{LL}$. Since Equation (3) implies $(d_j + p_{jj} - Rp_{ij})(d_j + p_{j\kappa(j)} - Rp_{ij}) \leq 0$ for $i, j = L, H$, we get $0 \leq d_L + p_{LL} - Rp_{LL}$ and thus $\frac{d_L}{r} = p_{LL} = p_{LH}$ because of Equation (15) and $p_{LH} \leq p_{LL}$. But from Equations (4) and (15) we get $p_{LH} \geq R^{-1}(d_H + \frac{d_L}{r}) > \frac{d_L}{r} = p_{LL}$, contradicting $p_{LH} \leq p_{LL}$. Therefore $p_{LH} > p_{LL}$. Similarly $p_{HH} > p_{HL}$ follows. This proves Equation (8). Because of Equation (3), $d_j + p_{jj} - Rp_{jj} = 0$ implies $d_j + p_{j\kappa(j)} - Rp_{jj} = 0$ and thus $p_{j\kappa(j)} = p_{jj}$ for $j = L, H$. This gives strict inequalities in Equation (15) and thus Equation (7). From $p_{HH} > p_{HL}$ and Equation (3) we derive $0 > d_H - rp_{LH} + p_{HL} - p_{LH}$ and Equation (9) follows because of Equation (7). Finally, the transition matrix $\Phi = (\phi_{ij})$ for the states $(p_{LL},\ p_{HL},\ p_{LH},\ p_{HH})$ is given by

$$\Phi \quad = \quad \begin{bmatrix} \pi & 0 & 1-\pi & 0 \\ \pi & 0 & 1-\pi & 0 \\ 0 & 1-\pi & 0 & \pi \\ 0 & 1-\pi & 0 & \pi \end{bmatrix}.$$

Since $\Phi^2 \gg 0$ and $\phi_{ii} > 0$ for $i = 1, 4$, Φ is indecomposable (irreducible) and primitive. This implies ergodicity of $\{P_t\}_{t=1}^{\infty}$ [see, e.g., Cox and Miller (1965, p. 124)]. The limiting probabilities follow from $(\frac{\pi}{2}, \frac{1-\pi}{2}, \frac{1-\pi}{2}, \frac{\pi}{2})\Phi = (\frac{\pi}{2}, \frac{1-\pi}{2}, \frac{1-\pi}{2}, \frac{\pi}{2})$.

Proof of Proposition 2. Let $p = (p_{LL},\ p_{HL},\ p_{LH},\ p_{HH})$ be an equilibrium and let $m = (m_L,\ m_H)$ be the associated participation levels. Define $F_L(y,n) := \pi\ (d_L + p_{LL} - Ry)\ U'[Rw + (d_L + p_{LL} - Ry)\frac{S}{n}] + (1-\pi)\,(d_L + p_{LH} - Ry)\,U'[Rw + (d_L + p_{LH} - Ry)\frac{S}{n}]$, where $(y,n) \in \{(y,n) \mid y \geq 0,\ n > 0,\ Rw + (d_L + p_{Lj} - Ry)\frac{S}{n} > 0,\ j = L, H\}$. Because of Equation (3), $F_L(p_{LL},\ m_L) = 0$ and $F_L(p_{HL}, m_H) = 0$. For the partial derivatives of F_L we get

$$\begin{aligned} \frac{\partial F_L(y,n)}{\partial n} &= -\frac{S}{n^2} E \left\{ (d_L + \tilde{P}_{t+1} - Ry)^2 \right. \\ &\quad \left. \times\, U''\left[Rw + (d_L + \tilde{P}_{t+1} - Ry)\frac{S}{n} \right] \mid D_t = d_L \right\} > 0; \end{aligned} \tag{16}$$

$$\begin{aligned} \frac{\partial F_L(y,n)}{\partial y} &= -RE\left\{ U'\left[Rw + (d_L + \tilde{P}_{t+1} - Ry)\frac{S}{n} \right] \mid D_t = d_L \right\} \\ &\quad - R\frac{S}{n} E \left\{ (d_L + \tilde{P}_{t+1} - Ry) \right. \\ &\qquad \left. \times\, U''\left[Rw + (d_L + \tilde{P}_{t+1} - Ry)\frac{S}{n} \right] \mid D_t = d_L \right\}. \end{aligned} \tag{17}$$

Because of $\alpha = -U''/U'$, $E\{(d_L + \tilde{P}_{t+1} - Ry)U''[Rw + (d_L + \tilde{P}_{t+1} - Ry)\frac{S}{n}] \mid D_t = d_L\} = -E\{(d_L + \tilde{P}_{t+1} - Ry)U'[Rw + (d_L + \tilde{P}_{t+1} - Ry)\frac{S}{n}][\alpha[Rw + (d_L + \tilde{P}_{t+1} - Ry)\frac{S}{n}] - \alpha(Rw)] \mid D_t = d_L\} - \alpha(Rw)F_L(y,n)$. Since $\alpha' \leq 0$ (Assumption 3), $(d_L + \tilde{P}_{t+1} - Ry)\{\alpha[Rw + (d_L + \tilde{P}_{t+1} - Ry)\frac{S}{n}] - \alpha(Rw)] \leq 0$. Consequently $F_L(y,n) \leq 0$ implies $\frac{\partial F_L(y,n)}{\partial y} < 0$ because of Equation (17), and in particular $\frac{\partial F_L(y,n)}{\partial y} < 0$ for $F_L(y,n) = 0$. Because of this, Equation (16), and the implicit function theorem there exists a differentiable function $g(n)$ such that $F_L[g(n), n] = 0$ and $g'(n) > 0$

(since $y \in \mathbb{R}^1$ this holds globally). From this and $F_L(p_{LL}, m_L) = 0 = F_L(p_{HL}, m_H)$ we get the result that $p_{HL} \geq p_{LL}$ if and only if $m_H \geq m_L$, and $p_{HL} > p_{LL}$ if and only if $m_H > m_L$. Analogously we can derive that $p_{HH} \geq p_{LH}$ ($p_{HH} > p_{LH}$) if and only if $m_H \geq m_L$ ($m_H > m_L$).

Proof of Proposition 3. Define the sets $\mathcal{P}$ and $\mathcal{M}$ by $\mathcal{P} := [\frac{d_L}{r}, \frac{d_H}{r}]^4$ and $\mathcal{M} := [\varepsilon, 1]^2$, respectively. Let $p = (p_{LL},\ p_{HL},\ p_{LH},\ p_{HH})$ denote an element of $\mathcal{P}$, and $m = (m_L, m_H)$ an element of $\mathcal{M}$. As in the proof of Proposition 1 let $\kappa(j)$, $j \in \{L, H\}$ be defined by $\kappa(L) = H$ and $\kappa(H) = L$. Further, define $c_{ijj}(p, m)$ and $c_{ij\kappa(j)}(p, m)$ by $c_{ijj}(p, m) := Rw + (d_j + p_{jj} - Rp_{ij})\frac{S}{m_i}$ and $c_{ij\kappa(j)}(p, m) := Rw + (d_j + p_{j\kappa(j)} - Rp_{ij})\frac{S}{m_i}$, respectively, where $i, j = L, H$. In the following we suppress the arguments (p, m) and just write c_{ijj} and $c_{ij\kappa(j)}$, respectively. Corresponding to Equation (4), we define four functions $f_{ij} : \mathcal{P} \times \mathcal{M} \to [\frac{d_L}{r}, \frac{d_H}{r}]$, $i = L, H$, by

$$f_{ij}(p, m) := \begin{cases} \frac{1}{R}\,\frac{(d_j + p_{jj})\pi U'(c_{ijj}) + (d_j + p_{j\kappa(j)})(1-\pi)U'(c_{ij\kappa(j)})}{\pi U'(c_{ijj}) + (1-\pi)U'(c_{ij\kappa(j)})} & \text{if} \\ \qquad c_{ijj} > 0 \text{ and } c_{ij\kappa(j)} > 0, \\ \frac{1}{R}\min\left[d_j + p_{jj}, d_j + p_{j\kappa(j)}\right] & \text{otherwise,} \end{cases} \tag{18}$$

for $i, j = L, H$.

First, we show that $f_{ij}(p, m) \in [\frac{d_L}{r}, \frac{d_H}{r}]$. This follows from $f_{ij}(p, m) \geq R^{-1}\min[d_j + p_{jj}, d_j + p_{j\kappa(j)}] \geq \frac{d_L}{r}$, and $f_{ij}(p, m) \leq R^{-1}\max[d_j + p_{jj}, d_j + p_{j\kappa(j)}] \leq \frac{d_H}{r}$. Next, we prove that $f_{ij}(p, m)$ is continuous. For $(p, m) \in \mathcal{P} \times \mathcal{M}$ such that (i) $c_{ijj} < 0$, or (ii) $c_{ij\kappa(j)} < 0$, or (iii) $c_{ijj} > 0$ and $c_{ij\kappa(j)} > 0$ continuity is obvious. This also covers the case where c_{ijj} and $c_{ij\kappa(j)}$ are both negative. The remaining cases are $(p, m) \in \mathcal{P} \times \mathcal{M}$ such that $c_{ijj} \geq 0$ and $c_{ij\kappa(j)} \geq 0$ with at least one equality holding. Take the case where i and j are fixed and $c_{ijj} = 0$, $c_{ij\kappa(j)} > 0$ for $(\hat{p}, \hat{m}) \in \mathcal{P} \times \mathcal{M}$. Consequently, $f_{ij}(\hat{p}, \hat{m}) = R^{-1}\min[d_j + \hat{p}_{jj}, d_j + \hat{p}_{j\kappa(j)}] = R^{-1}(d_j + \hat{p}_{jj})$ because $0 < c_{ij\kappa(j)} - c_{ijj} = (\hat{p}_{j\kappa(j)} - \hat{p}_{jj})\frac{S}{m_i}$ and hence $\hat{p}_{j\kappa(j)} > \hat{p}_{jj}$. Since $\lim_{c \to 0} U' = \infty$, $\lim_{(p,m)\to(\hat{p},\hat{m})} f_{ij}(p, m) = R^{-1}(d_j + \hat{p}_{jj}) = f_{ij}(\hat{p}, \hat{m})$. Thus f_{ij} is continuous at $(\hat{p}, \hat{m})$. Similarly, continuity follows at $(\hat{p}, \hat{m}) \in \mathcal{P} \times \mathcal{M}$ such that $c_{ijj} > 0$ and $c_{ij\kappa(j)} = 0$ for any fixed i and j because $f_{ij}(\hat{p}, \hat{m}) = R^{-1}\min[d_j + \hat{p}_{jj}, d_j + \hat{p}_{j\kappa(j)}] = d_j + \hat{p}_{j\kappa(j)} = \lim_{(p,m)\to(\hat{p},\hat{m})} f_{ij}(p, m)$. The last case is $(\hat{p}, \hat{m}) \in \mathcal{P} \times \mathcal{M}$ such that $c_{ijj} = c_{ij\kappa(j)} = 0$. This implies $\hat{p}_{j\kappa(j)} = \hat{p}_{jj}$ and $f_{ij}(\hat{p}, \hat{m}) = R^{-1}(d_j + \hat{p}_{jj}) = R^{-1}(d_j + \hat{p}_{j\kappa(j)})$. Let $\{(p^s, m^s)\}_{s=1}^{\infty}$ be a sequence of vectors $(p^s_{LL},\ p^s_{HL},\ p^s_{LH},\ p^s_{HH},\ m^s_L,\ m^s_H) \in \mathcal{P} \times \mathcal{M}$ which converges to $(\hat{p}, \hat{m})$. It is sufficient to consider sequences for which the two conditions $c_{ijj}(p^s, m^s) > 0$ and $c_{ij\kappa(j)}(p^s, m^s) > 0$ are satisfied either for all

$s = 1, 2, \ldots$ or for no $s = 1, 2, \ldots$. In the second case, $f_{ij}(p^s, m^s) = R^{-1}\min[d_j + p^s_{jj}, d_j + p^s_{j\kappa(j)}]$ and $\lim_{s\to\infty} f_{ij}(p^s, m^s) = R^{-1}(d_j + \hat{p}_{jj}) = f_{ij}(\hat{p}, \hat{m})$. In the first case, $\lim_{j\to\infty} f_{ij}(p^s, m^s) = R^{-1}\lim_{s\to\infty}\{(d_j + p^s_{jj}) + [(d_j + p^s_{j\kappa(j)}) - (d_j + p^s_{jj})]\frac{(1-\pi)U'[c_{ij\kappa(j)}(p^s, m^s)]}{\pi\ U'[c_{ijj}(p^s, m^s)] + (1-\pi)U'[c_{ij\kappa(j)}(p^s, m^s)]} = R^{-1}(d_j + \hat{p}_{jj}) = f_{ij}(\hat{p}, \hat{m})$ because $\lim_{s\to\infty}[p^s_{j\kappa(j)} - p^s_{jj}] = 0$ and $0 \leq \frac{(1-\pi)U'(c_{ij\kappa(j)})}{\pi U'(c_{ijj}) + (1-\pi)U'(c_{ij\kappa(j)})} \leq 1$. Thus $\lim_{(p,m)\to(\hat{p},\hat{m})} f_{ij}(p, m) = f_{ij}(\hat{p}, \hat{m})$ and continuity follows.

For the subsequent analysis we extend the function U to the negative reals by defining $U(c) = U(0)$ for any $c < 0$, and we define the function u_L: $\mathcal{P} \times \mathcal{M} \to \mathbb{R}$ by

$$\begin{aligned} u_L(p, m) \ :=\ & \pi^2 U(c_{LLL}) + \pi(1-\pi)U(c_{LLH}) + \pi(1-\pi)U(c_{LHH}) \\ & + (1-\pi)^2 U(c_{LHL}) - U(Rw). \end{aligned}$$

Let the function g_L: $\mathcal{P} \times \mathcal{M} \to [\varepsilon, 1]$ be defined as follows:

$$g_L(p, m) := \begin{cases} \varepsilon \quad \text{if} \quad u_L(p, m) < 0 \\ k^{-1}[u_L(p, m)] \quad \text{if} \quad u_L(p, m) \geq 0, \end{cases} \tag{19}$$

where k^{-1}: $\mathbb{R}_+ \to [\varepsilon, 1]$ is the inverse function of $k(a)$, $a \geq \varepsilon$. By Assumption 4 $g_L(p, m)$ is continuous. Similar to u_L and g_L we define the functions u_H: $\mathcal{P} \times \mathcal{M} \to \mathbb{R}$ and g_H: $\mathcal{P} \times \mathcal{M} \to [\varepsilon, 1]$ by

$$\begin{aligned} u_H(p, m) \ :=\ & \pi^2 U(c_{HHH}) + \pi(1-\pi)U(c_{HHL}) + \pi(1-\pi)U(c_{HLL}) \\ & + (1-\pi)^2 U(c_{HLH}) - U(Rw) \end{aligned}$$

and

$$g_H(p, m) := \begin{cases} \varepsilon \quad \text{if} \quad u_H(p, m) < 0 \\ k^{-1}[u_H(p, m)] \quad \text{if} \quad u_H(p, m) \geq 0 \end{cases} \tag{20}$$

Again, g_H is continuous.

The functions $(f_{LL},\ f_{HL},\ f_{LH},\ f_{HH},\ g_L,\ g_H)$ continuously map the compact and convex set $\mathcal{P} \times \mathcal{M}$ into itself. By Brouwer's fixed point theorem there exists a fixed point $(p^*, m^*) = (p^*_{LL}, p^*_{HL}, p^*_{LH}, p^*_{HH}, m^*_L, m^*_H)$, that is, $p^*_{ij} = f_{ij}(p^*, m^*)$ and $m^*_i = g_i(p^*, m^*)$, $i, j = L, H$. The last step of the proof of Proposition 3 is to show that (p^*, m^*) constitutes an equilibrium. First, we prove that $c_{ijj}(p^*, m^*) > 0$ and $c_{ij\kappa(j)}(p^*, m^*) > 0$. Assume $Rw + (d_j + p^*_{jh} - Rp^*_{ij})\frac{S}{m^*_i} \leq 0$ for some fixed triple $(i, j, h) \in \{L, H\}^3$. Then $p^*_{ij} = f_{ij}(p^*, m^*) = R^{-1}\min(d_j + p^*_{jL}, d_j + p^*_{jH}) \leq R^{-1}(d_j + p^*_{jh})$ because of Equation (18). Therefore $0 \leq d_j + p^*_{jh} - Rp^*_{ij}$ contradicting $Rw + (d_j + p^*_{jh} - Rp^*_{ij})\frac{S}{m_i} \leq 0$ because $Rw > 0$. Consequently, $c_{ijj}(p^*, m^*) > 0$ and $c_{ij\kappa(j)}(p^*, m^*) > 0$, and thus (p^*, m^*) satisfies Equation (4).

Finally, we have to show that $u_i(p^*, m^*) \geq 0$, $i = L, H$. Let $\{\tilde{P}_t^*\}_{t=1}^{\infty}$ be the prices associated with p^*. Since Equation (4) holds, $E\{U[Rw + (D_t + \tilde{P}_{t+1}^* - RP_t^*)x] \mid D_t\}$ is maximized at $x = \frac{S}{m_L^*}$ if $P_t^* = p_{Lj}^*$, and at $x = \frac{S}{m_H^*}$ if $P_t^* = p_{Hj}^*$, $j = L, H$. Because of $U'' < 0$ the maximum is unique. Therefore $\{U[Rw + (D_t + \tilde{P}_{t+1}^* - RP_t^*)\frac{S}{n_t^*}] \mid D_t\} > U(Rw)$, where $n_t^* = m_L^*$ if $P_t^* = P_{Lj}^*$ and $n_t^* = m_H^*$ if $P_t^* = p_{Hj}^*$, $j = L, H$. Consequently, $E\{U[Rw + (D_t + \tilde{P}_{t+1}^* - RP_t^*)\frac{S}{n_t^*}] \mid D_{t-1}\} - U(Rw) > 0$. Since the left-hand side of the last inequality is $u_L(p^*, m^*)$ for $D_{t-1} = d_L$ and $u_H(p^*, m^*)$ for $D_{t-1} = d_H$, $u_L(p^*, m^*) > 0$ and $u_H(p^*, m^*) > 0$. This implies that (p^*, m^*) satisfies Equations (5) and (6), and concludes the proof of Proposition 3. It also implies $m_i^* = g_i(p^*, m^*) > \varepsilon$ for $i = L, H$, that is, Corollary 1.

Proof of Lemma 1. For notational convenience we normalize $\frac{S}{M} = 1$. Since M is constant, this can be done without loss of generality by choosing an appropriate new unit for shares. Next we simplify the notation by defining

$$\begin{aligned} v_{ij} &:= d_i + p_j - Rp_i, \\ c_{ij} &:= Rw + v_{ij}, \\ U_{ij} &:= U(c_{ij}), \\ U'_{ij} &:= U'(c_{ij}), \end{aligned}$$

where in all cases $i, j = L, H$. Optimization and market clearing imply that Equation (3) holds with $p_{HL} = p_{LL} = p_L$, $p_{LH} = p_{HH} = p_H$, $m_L = m_H = M$, and $\frac{S}{M} = 1$, that is,

$$\pi v_{LL} U'_{LL} + (1 - \pi) v_{LH} U'_{LH} = 0, \tag{21}$$

$$\pi v_{HH} U'_{HH} + (1 - \pi) v_{HL} U'_{HL} = 0. \tag{22}$$

Analogously to Proposition 1 we get $p_H > p_L$. Together with Equations (21) and (22) this implies $v_{LL} < 0$, $v_{LH} > 0$, $v_{HH} > 0$, and $v_{HL} < 0$. We want to prove that $\Delta U > 0$, where ΔU is defined as

$$\begin{aligned} \Delta U &:= E[U(\tilde{c}_{t+1}) \mid D_{t-1} = d_H] - E[U(\tilde{c}_{t+1}) \mid D_{t-1} = d_L] \\ &= (2\pi - 1)\{[\pi U_{HH} + (1 - \pi) U_{HL}] - [\pi U_{LL} + (1 - \pi) U_{LH}]\}. \end{aligned}$$

In order to derive $\Delta U > 0$ we first have to prove a series of technical claims.

Claim 1. $v_{HL} < v_{LL} < 0 < v_{HH} < v_{LH}$.

Proof. Define the function $F(y)$ by $F(y) := \pi(d_L + p_L - y)U'(Rw + d_L + p_L - y) + (1 - \pi)(d_L + p_H - y)U'(Rw + d_L + p_H - y)$. From

Equations (21) and Equation (22) we get $F(Rp_L) = 0$ and $F(Rp_H + d_L - d_H) = \pi v_{HL} U'_{HL} + (1-\pi) v_{HH} U'_{HH} < 0$ because of Equation (22), $v_{HL} < 0$, $v_{HH} > 0$ and $\pi > 1/2$. Further, $\partial F(y)/\partial y = -[\pi U'(Rw + d_L + p_L - y) + (1-\pi) U'(Rw + d_L + p_H - y)] + \pi(d_L + p_L - y)[\alpha(Rw + d_L + p_L - y) - \alpha(Rw)]U'(Rw + d_L + p_L - y) + (1-\pi)(d_L + p_H - y)[\alpha(Rw + d_L + p_H - y) - \alpha(Rw)]U'(Rw + d_L + p_H - y) + \alpha(Rw)F(y) < 0$ for $F(y) \leq 0$ because absolute risk aversion $\alpha(c)$ is nonincreasing (this is analogous to the part of the proof of Proposition 2 where we showed that $F_L(y, n) \leq 0$ implies $\partial F_L(y, n)/\partial y < 0$). Thus $F(Rp_L) = 0$ and $F(Rp_H + d_L - d_H) < 0$ imply $Rp_H + d_L - d_H > Rp_L$ and thus $0 < R(p_H - p_L) - (d_H - d_L) = v_{LL} - v_{HL} = v_{LH} - v_{HH}$. This proves Claim 1. ■

Claim 2. $v_{HH} + v_{LL} = v_{LH} + v_{HL} > 0$.

Proof. Obviously $v_{HH} + v_{LL} = v_{LH} + v_{HL} = d_H + d_L - r(p_H + p_L)$. Because of $U''(c) < 0$, Equations (21) and (22) imply $Rp_L < d_L + (1-\pi)p_H + \pi p_L$ and $Rp_H < d_H + \pi p_H + (1-\pi)p_L$. Adding the inequalities gives $0 < d_H + d_L - r(p_H + p_L)$ and Claim 2 follows. ■

Claim 3. $\frac{-v_{HL}}{-v_{LL}} > \frac{v_{LH} - v_{HL}}{v_{HH} - v_{LL}}$.

Proof. Because of Claim 2, $\frac{v_{LH}}{v_{HH}} = \frac{-v_{HL} + \varphi}{-v_{LL} + \varphi}$ where $\varphi := v_{HL} + v_{LH} = v_{LL} + v_{HH} > 0$. Thus, $\frac{v_{LH}}{v_{HH}} < \frac{-v_{HL}}{-v_{LL}}$ because $-v_{HL} > -v_{LL} > 0$ (Claim 1). This implies $\frac{v_{LH} - v_{HL}}{v_{HH} - v_{LL}} = \frac{v_{LH}}{v_{HH}} \frac{v_{HH}}{v_{HH} - v_{LL}} + \frac{(-v_{HL})}{(-v_{LL})} \frac{(-v_{LL})}{v_{HH} - v_{LL}} < \frac{-v_{HL}}{-v_{LL}}$. ■

Claim 4. $\frac{\pi}{1-\pi} > \frac{v_{LH} - v_{HL}}{v_{HH} - v_{LL}}$.

Proof. Since $0 \geq \alpha'(c) = \frac{1}{[U'(c)]^2} \left\{ [U''(c)]^2 - U'(c)U'''(c) \right\}$,

$$U''' > 0. \tag{23}$$

Define $\gamma := v_{LH} - v_{HH} = v_{LL} - v_{HL} > 0$ (Claim 1), $\varphi := \frac{1}{\gamma}(U'_{HL} - U'_{LL}) > 0$ and $\psi := \frac{1}{\gamma}(U'_{HH} - U'_{LH}) > 0$. Claim 1 and Equation (23) imply $\varphi > \psi > 0$. Because of Equations (21) and (22),

$$\frac{\pi}{(1-\pi)} = \frac{v_{LH} U'_{LH} - v_{HL} U'_{HL}}{v_{HH} U'_{HH} - v_{LL} U'_{LL}}. \tag{24}$$

Rearranging the denominator on the right-hand side of Equation (24) and using $\varphi > \psi > 0$ gives $v_{HH} U'_{HH} - v_{LL} U'_{LL} < (v_{HH} - v_{LL})(U'_{LH} - \psi v_{HL}) - v_{LL}[U'_{HL} - U'_{LH} + \psi(v_{HL} - v_{LH})]$. Moreover, Equation (23) implies $\frac{U'_{HL} - U'_{LH}}{v_{LH} - v_{HL}} > \frac{U'_{HH} - U'_{LH}}{v_{LH} - v_{HH}} = \psi$ and thus $U'_{HL} - U'_{LH} + \psi(v_{HL} - v_{LH}) > 0$. Claim 3 implies $\frac{-v_{HL}}{v_{LH} - v_{HL}} > \frac{-v_{LL}}{v_{HH} - v_{LL}}$. From this we get, after rearranging Equation (24), $\frac{\pi}{1-\pi} > \frac{(v_{LH}-v_{HL})(U'_{LH}-\psi v_{HL})-v_{HL}[U'_{HL}-U'_{LH}+\psi(v_{HL}-v_{LH})]}{(v_{HH}-v_{LL})(U'_{LH}-\psi v_{HL})-v_{LL}[U'_{HL}-U'_{LH}+\psi(v_{HL}-v_{LH})]} > \frac{v_{LH}-v_{HL}}{v_{HH}-v_{LL}}$. ■

Claim 5. $(1-\pi)v_{HL}^2 - \pi v_{LL}^2 > (1-\pi)v_{LH}^2 - \pi v_{HH}^2$.

Proof. Claims 2 and 4 imply $\pi(v_{HH}^2 - v_{LL}^2) > (1-\pi)(v_{LH}^2 - v_{HL}^2)$ and thus Claim 5. ■

With the help of the preceding claims we can now prove the lemma, that is, $\Delta U > 0$. Although in equilibrium each young agent has to hold exactly $\frac{S}{M} = 1$ shares, in the following it will be useful to consider portfolios consisting of $x \geq 0$ shares and to analyze how expected utility varies with x. We define $W_H(x)$ and $W_L(x)$, $x \in \mathbb{R}$ by $W_H(x) := (1-\pi)U(Rw + v_{HL}x) + \pi U(Rw + v_{HH}x)$ and $W_L(x) := \pi U(Rw + v_{LL}x) + (1-\pi)U(Rw + v_{LH}x)$. Differentiating we get $W'_H(x) - W'_L(x) = (1-\pi)v_{HL}U'(Rw + v_{HL}x) + \pi v_{HH}U'(Rw + v_{HH}x) - \pi v_{LL}U'(Rw + v_{LL}x) - (1-\pi)v_{LH}U'(Rw + v_{LH}x)$, and $W''_H(x) - W''_L(x) = (1-\pi)v_{HL}^2 U''(Rw + v_{HL}x) + \pi v_{HH}^2 U''(Rw + v_{HH}x) - \pi v_{LL}^2 U''(Rw + v_{LL}x) - (1-\pi)v_{LH}^2 U''(Rw + v_{LH}x)$. For the rest of the proof we assume $x > 0$. From Claim 1 and Equation (23) we get

$$\begin{aligned} U''(Rw + v_{HL}x) &< U''(Rw + v_{LL}x) \\ &< U''(Rw + v_{HH}x) < U''(Rw + v_{LH}x) < 0. \end{aligned} \tag{25}$$

Claim 5 and Equation (25) give, after some calculation, $(1-\pi)v_{HL}^2 U''(Rw + v_{HL}x) - \pi v_{LL}^2 U''(Rw + v_{LL}x) < (1-\pi)v_{LH}^2 U''(Rw + v_{LH}x) - \pi v_{HH}^2 U''(Rw + v_{HH}x)$. This implies $W''_H(x) - W''_L(x) < 0$. We have $W'_H(1) - W'_L(1) = 0$ because of Equations (21) and (22). Consequently, $\bar{x} = 1$ maximizes $W_H(x) - W_L(x)$. Since $W_H(0) - W_L(0) = 0$ and $W''_H(x) - W''_L(x) < 0$ for $x > 0$, $W_H(1) > W_L(1)$, that is, $\pi U_{HH} + (1-\pi)U_{HL} > \pi U_{LL} + (1-\pi)U_{LH}$. This implies $\Delta U > 0$ and thus concludes the proof of the lemma.

Proof of Remark 1. Expected utility of the sum of any given deterministic consumption $c \geq \pi(p_H - p_L)$ plus the payoff of $x > 0$ units (i) of lottery $\hat{\mathcal{L}}_H$, and (ii) of lottery $\hat{\mathcal{L}}_L$ is given (i) by $V_H(x) = (1-\pi)U[c - \pi(p_H - p_L)x] + \pi U[c + (1-\pi)(p_H - p_L)x]$, and (ii) by $V_L(x) = \pi U[c - (1-\pi)(p_H - p_L)x] + (1-\pi)U[c + \pi(p_H - p_L)x]$. Since both lotteries have zero expected payoffs, x units of $\hat{\mathcal{L}}_H$ are more risky than x units of $\hat{\mathcal{L}}_L$, if and only if $V_H(x) < V_L(x)$. The derivatives of $V_H(x)$ and $V_L(x)$ are $V'_H(x) = -\pi(1-\pi)(p_H - p_L)U'[c - \pi(p_H - p_L)x] + \pi(1-\pi)(p_H - p_L)U'[c + (1-\pi)(p_H - p_L)x] < 0$ and $V'_L(x) = -\pi(1-\pi)(p_H - p_L)U'[c - (1-\pi)(p_H - p_L)x] + \pi(1-\pi)(p_H - p_L)U'[c + \pi(p_H - p_L)x] < 0$, respectively. For $x > 0$ this gives $V'_L(x) - V'_H(x) = \pi(1-\pi)(p_H - p_L)[\{U'[c - \pi(p_H - p_L)x] - U'[c - (1-\pi)(p_H - p_L)x]\} - \{U'[c + (1-$

$\pi)(p_H - p_L)x] - U'[c + \pi(p_H - p_L)x]\}] > 0$ because of Equation (23). Therefore $V_H'(x) < V_L'(x) < 0$ for all $x \geq 0$. Moreover, $V_H(0) = V_L(0)$. From this the remark follows.

Proof of Proposition 4. Let the sets $\mathcal{P}$ and $\mathcal{M}$ with elements p and m, respectively, be defined as in the proof of Proposition 3, and let the functions $f_{ij}(p,m)$ and $g_i(p,m)$, $i, j = L, H$ be given by Equations (18)–(20). Further, define $G_H : \mathcal{P} \times \mathcal{M} \to [\varepsilon, 1]$ by

$$G_H(p,m) := \max[g_L(p,m), g_H(p,m)]. \tag{26}$$

The functions $(f_{LL},\ f_{HL},\ f_{LH},\ f_{HH},\ g_L,\ G_H)$ continuously map the compact and convex set $\mathcal{P} \times \mathcal{M}$ into itself. By Brouwer's fixed point theorem there exists a fixed point $(p^*, m^*) = (p^*_{LL}, p^*_{HL}, p^*_{LH}, p^*_{HH}, m^*_L, m^*_H)$. As in the proof of Proposition 3, $Rw + (d_j + p^*_{jh} - Rp^*_{ij})\frac{S}{m^*_i} > 0$ for $i, j, h = L, H$; and $u_L(p^*, m^*) > 0$, where $u_L : \mathcal{P} \times \mathcal{M} \to \mathbb{R}$ is defined as in the proof of Proposition 3 [just before Equation (19)]. By contradiction we prove that $m^*_H > m^*_L$. Assume $m^*_H \leq m^*_L$. This implies $m^*_H = m^*_L$ since $m^*_H = G_H(p^*, m^*) \geq g_L(p^*, m^*) = m^*_L$. Because of this and Lemma 1, $u_H(p^*, m^*) > u_L(p^*, m^*) > 0$, where $u_H : \mathcal{P} \times \mathcal{M} \to \mathbb{R}$ is defined as in the proof of Proposition 3 [just before Equation (20)]. Therefore $g_H(p^*, m^*) > g_L(p^*, m^*) = m^*_L$. But this implies $m^*_H = G_H(p^*, m^*) = g_H(p^*, m^*) > g_L(p^*, m^*) = m^*_L$, which contradicts our assumption $m^*_H \leq m^*_L$. This proves $m^*_H > m^*_L$, which also implies $u_H(p^*, m^*) > u_L(p^*, m^*) > 0$. Finally, $p^*_{HH} > p^*_{LH} > p^*_{HL} > p^*_{LL}$ follows from $m^*_H > m^*_L$ and Propositions 1 and 2.

Proof of Lemma 2. Let $P = (P_{LL}, P_{HL}, P_{LH}, P_{HH}) \in [\frac{d_L}{r}, \frac{d_H}{r}]^4$ denote any vector of share prices indexed by the level of past and present dividends, and let, for this proof, $M = (M_L, M_H) \in [\varepsilon, 1]^2$ denote any vector of participation levels indexed by the level of past dividends. As in the proof of Proposition 1, let $\kappa(j)$, $j \in \{L, H\}$ be defined by $\kappa(L) = H$ and $\kappa(H) = L$. Corresponding to the market clearing condition Equation (3) we define

$$\begin{aligned} F_{ij}(P,M) \;:=\; & \pi\ (d_j + P_{jj} - RP_{ij})\, U'\left[Rw + (d_j + P_{jj} - RP_{ij})\frac{S}{M_i}\right] \\ & + (1-\pi)\ (d_j + P_{j\kappa(j)} - RP_{ij}) \\ & \times U'\left[Rw + (d_j + P_{j\kappa(j)} - RP_{ij})\frac{S}{M_i}\right], \\ & i, j = L, H; \quad \kappa(L) = H, \quad \kappa(H) = L. \end{aligned} \tag{27}$$

Because of $d_j + P_{jj} - RP_{ij} \neq d_j + P_{j\kappa(j)} - RP_{ij}$, differentiation of F_{ij}

gives

$$\frac{\partial F_{Lj}}{\partial M_H} = \frac{\partial F_{Hj}}{\partial M_L} = 0, \qquad j = L, H, \tag{28}$$

$$\frac{\partial F_{ij}}{\partial M_i} > 0 \qquad \text{for} \quad P_{jj} \neq P_{j\kappa(j)}, \qquad i, j = L, H. \tag{29}$$

Let Γ denote the matrix

$$\Gamma = \begin{bmatrix} \frac{\partial F_{LL}}{\partial P_{LL}} & \frac{\partial F_{HL}}{\partial P_{LL}} & \frac{\partial F_{LH}}{\partial P_{LL}} & \frac{\partial F_{HH}}{\partial P_{LL}} \\ \frac{\partial F_{LL}}{\partial P_{HL}} & \frac{\partial F_{HL}}{\partial P_{HL}} & \frac{\partial F_{LH}}{\partial P_{HL}} & \frac{\partial F_{HH}}{\partial P_{HL}} \\ \frac{\partial F_{LL}}{\partial P_{LH}} & \frac{\partial F_{HL}}{\partial P_{LH}} & \frac{\partial F_{LH}}{\partial P_{LH}} & \frac{\partial F_{HH}}{\partial P_{LH}} \\ \frac{\partial F_{LL}}{\partial P_{HH}} & \frac{\partial F_{HL}}{\partial P_{HH}} & \frac{\partial F_{LH}}{\partial P_{HH}} & \frac{\partial F_{HH}}{\partial P_{HH}} \end{bmatrix} = (\gamma_{ij}); \qquad i, j = 1, \ldots, 4.$$

We show that Equation (11) implies that Γ has a dominant diagonal, that is, that $|\gamma_{jj}| > \sum_{i \neq j} |\gamma_{ij}|$ for all $j = 1, \ldots, 4$. First, consider $\partial F_{HH}/\partial P_{ij}$. Applying Equation (11) it is easy to see that $\frac{\partial F_{HH}}{\partial P_{LL}} = \frac{\partial F_{HH}}{\partial P_{LH}} = 0$, $\frac{\partial F_{HH}}{\partial P_{HL}} > 0$, and $\frac{\partial F_{HH}}{\partial P_{HH}} < -\frac{\partial F_{HH}}{\partial P_{HL}}$. Therefore $|\gamma_{44}| > \sum_{i \neq 4} |\gamma_{i4}|$. Similarly, we get $|\gamma_{11}| > \sum_{i \neq 1} |\gamma_{i1}|$, $|\gamma_{22}| > \sum_{i \neq 2} |\gamma_{i2}|$, and $|\gamma_{33}| > \sum_{i \neq 3} |\gamma_{i3}|$. This proves that Γ has diagonal dominance. Further, the proof shows that $\gamma_{ij} \geq 0$ for $i \neq j$ and $\gamma_{ii} < 0$, $i = 1, \ldots, 4$. Together with diagonal dominance this implies that there exists a $q \in \mathbb{R}^4_+$ such that $-\Gamma q \gg 0$ [see, e.g., Takayama (1974, Theorem 4.C.3, p. 382)]. Let ρ be defined by $\rho := \max_i |\gamma_{ii}| = \max_i(-\gamma_{ii})$, thus $\rho \geq -\gamma_{ii}$ for all $i = 1, \ldots, 4$. Define the matrix $\Psi = (\psi_{ij})$ by $\Psi := \rho I + \Gamma$. From $\psi_{ij} = \gamma_{ij} \geq 0$ for $i \neq j$ and $\psi_{ii} = \rho + \gamma_{ii} \geq 0$, where $i, j = 1, \ldots, 4$, we get $\Psi \geq 0$. Further, Ψ is indecomposable since all elements of Γ except $\gamma_{21}, \gamma_{41}, \gamma_{42}, \gamma_{13}, \gamma_{14}$, and γ_{34} are nonzero. Therefore we have $-\Gamma = \rho I - \Psi$ where Ψ is nonnegative and indecomposable, and in addition we know that there exists a $q \geq 0$ such that $-\Gamma q \gg 0$. Consequently, Γ is nonsingular and

$$\Gamma^{-1} \ll 0 \tag{30}$$

[see, e.g., Takayama (1974, Theorem 4.C.9, p. 387)]. Next we observe that $F_{ij}(P, M) = 0$, $i, j = L, H$ has a positive solution $P^* \gg 0$ for each pair $M \in (0, 1]^2$. This follows from a fixed point argument as in the proof of Proposition 3, but with M given [i.e., for the functions f_{ij}, $i, j = L, H$ defined as in Equation (18), but with $m = M$ fixed, there exists a fixed point $P^* = [f_{LL}(P^*, M),\ f_{HL}(P^*, M),\ f_{LH}(P^*, M),\ f_{HH}(P^*, M)] \in [\frac{d_L}{r}, \frac{d_H}{r}]^4$ and it holds that $Rw + (d_j + P^*_{jh} - RP^*_{ij})\frac{S}{M_i} > 0$ for all $i, j, h = L, H$, and consequently $F_{ij}(P^*, M) = 0$ for all $i, j = L, H$].

Next we prove by contradiction that the solution P^* of $F_{ij}(P, M) = 0$, $i, j = L, H$ is unique for each pair $M \in (0, 1]^2$. First we define $V(z) :=$

$zU'(Rw+z)$ where $z \in [-Rw, \infty)$. Then $V'(z) > 0$ for $z > -Rw$ because of Equation (11) and $V'(-Rw) = U'(0) - RwU''(0) > 0$; thus $V'(z) > 0$ for all $z \in [-Rw, \infty)$. Assume now that $F_{ij}(P, M) = 0$, $i, j = L, H$ has two nonidentical solutions P^* and $P^{**} \neq P^*$ for some given M. Since $P^{**} \neq P^*$ there exist indices k and h such that $|P^*_{kh} - P^{**}_{kh}| > 0$ maximizes $|P^*_{ij} - P^{**}_{ij}|$ for $i, j = L, H$. Without loss of generality we can assume $P^*_{kh} > P^{**}_{kh}$, thus $P^*_{kh} - P^{**}_{kh} \geq P^*_{ij} - P^{**}_{ij}$ for $i, j = L, H$. Because of $r > 0$ and $V'(z) > 0$ this implies $F_{kh}(P^{**}, M) > F_{kh}(P^*, M)$, contradicting the assumption $F_{kh}(P^{**}, M) = 0 = F_{kh}(P^*, M)$. This contradiction proves the uniqueness of P^*.

Finally, because Γ is nonsingular and $F_{ij}(P^*, M) = 0$ for $i, j = L, H$, we can apply the implicit function theorem. Together with uniqueness of P^* this theorem implies Lemma 2: P^* is a differentiable function of M, and $\frac{\partial P^*_{ij}}{\partial M_k} > 0$ for all $i, j, k = L, H$. The second part follows from Equations (28)–(30) and $P^*_{iH} > P^*_{iL}$ for $i = L, H$.

Proof of Proposition 5. The proof is by contradiction. Assume that there exists an equilibrium with $m_L \geq m_H$. This implies $m_L > m_H$ because of Corollary 2, and thus Proposition 2 gives

$$p_{Lj} > p_{Hj}, \qquad j = L, H. \tag{31}$$

We simplify by defining for $i, j, k = L, H$:

$$\begin{aligned} v_{kij} &:= d_i + p_{ij} - Rp_{ki}, \\ c_{kij} &:= Rw + v_{kij}\frac{S}{m_k}, \\ U_{kij} &:= U(c_{kij}), \\ U'_{kij} &:= U'(c_{kij}), \end{aligned}$$

where $c_{kij} > 0$ since they are equilibrium values. Because of Equation (31),

$$v_{Hij} > v_{Lij} \qquad \text{for } i, j = L, H. \tag{32}$$

If $\pi U_{LLL} + (1-\pi)U_{LLH} < \pi U_{LHH} + (1-\pi)U_{LHL}$, then the subsequent inequalities, explained below, follow:

$$\begin{aligned} &E\left[U(\tilde{c}_{t+1})|D_{t-1} = d_L\right] \\ &\quad < (1-\pi)\left[\pi U\left(Rw + v_{HLL}\frac{S}{m_L}\right) + (1-\pi)U\left(Rw + v_{HLH}\frac{S}{m_L}\right)\right] \end{aligned}$$

$$+ \pi \left[\pi U\left(Rw + v_{HHH} \frac{S}{m_L} \right) + (1-\pi) U\left(Rw + v_{HHL} \frac{S}{m_L} \right) \right]$$
$$< E\left[U\left(\tilde{c}_{t+1} \right) | D_{t-1} = d_H \right],$$

where the first inequality is implied by $\pi > \frac{1}{2}$ and Equation (32), the second by the fact that in the situation where the equilibrium price is p_{Hi}, $i = L, H$, and demand $\frac{S}{m_H}$ maximizes expected utility whereas $\frac{S}{m_L}$ does not. Because of Assumption 4, $E[U(\tilde{c}_{t+1})|D_{t-1} = d_L] < E[U(\tilde{c}_{t+1})|D_{t-1} = d_H]$ contradicts $m_L > m_H$. Therefore

$$\pi U_{LLL} + (1-\pi) U_{LLH} \geq \pi U_{LHH} + (1-\pi) U_{LHL}. \tag{33}$$

Analogous to Lemma 1, we can normalize $\frac{S}{m_L} = 1$ without loss of generality by choosing an appropriate (new) unit for shares. We define δ by $\delta := \pi U_{HHH} + (1-\pi) U_{HHL} - [\pi U_{LLL} + (1-\pi) U_{LLH}]$. Because of Equation (3),

$$\delta = \pi \left[\int_{c_{LLL}}^{c_{HHH}} U'(c) dc + v_{LLL} U'_{LLL} - v_{HHH} \frac{S}{m_H} U'_{HHH} \right]$$
$$-(1-\pi) \left[\int_{c_{HHL}}^{c_{LLH}} U'(c) dc - v_{LLH} U'_{LLH} + v_{HHL} \frac{S}{m_H} U'_{HHL} \right]. \tag{34}$$

Let M, p_i, v_{ij}, c_{ij}, U_{ij}, U'_{ij} (where $i, j = L, H$) and ΔU be defined as in Lemma 1 and its proof. The normalizations $\frac{S}{m_L} = 1$ and $\frac{S}{M} = 1$ (proof of Lemma 1) give $m_L = M$. From Equation (12) and $m_H < m_L = M$ we get $p_{HH} < p_H$ and $p_{LL} < p_L$. Consequently, $v_{HHH} > v_{HH} > 0$ (see Claim 1, proof of Lemma 1). Similarly we get $0 > v_{LLL} > v_{LL}$, where $v_{LLL} < 0$ results from from Equations (3) and (8). Together with $\frac{S}{m_H} > \frac{S}{m_L} = 1$ this gives $c_{HHH} > c_{HH}$ and $c_{LLL} > c_{LL}$. Furthermore, $v_{HHH} \frac{S}{m_H} > v_{HH} > 0$ and Equation (11) imply $v_{HHH} \frac{S}{m_H} U'_{HHH} > v_{HH} U'_{HH}$. Because of thi and Equations (3) and (22) we get $-v_{HHL} \frac{S}{m_H} U'_{HHL} > -v_{HL} U'_{HL} > 0$, which in turn gives $v_{HHL} < 0$ and $c_{HHL} < c_{HL}$ because of $U'' < 0$. Analogously, $v_{LLH} > 0$ and $c_{LLH} < c_{LH}$ because of $0 > v_{LLL} > v_{LL}$. Thus $c_{HHL} < c_{HL} < Rw < c_{HH} < c_{HHH}$ and $c_{LL} < c_{LLL} < Rw < c_{LLH} < c_{LH}$. Therefore

$$\delta = -\pi \int_{c_{LLL}}^{Rw} \left[U'_{LLL} - U'(c) \right] dc + \pi \int_{Rw}^{c_{HHH}} \left[U'(c) - U'_{HHH} \right] dc$$
$$+ (1-\pi) \int_{c_{HHL}}^{Rw} \left[U'_{HHL} - U'(c) \right] dc$$
$$- (1-\pi) \int_{Rw}^{c_{LLH}} \left[U'(c) - U'_{LLH} \right] dc$$

$$
\begin{aligned}
&> -\pi \int_{c_{LL}}^{Rw} \left[U'_{LL} - U'(c)\right] dc + \pi \int_{Rw}^{c_{HH}} \left[U'(c) - U'_{HH}\right] dc \\
&\quad + (1-\pi) \int_{c_{HL}}^{Rw} \left[U'_{HL} - U'(c)\right] dc \\
&\quad - (1-\pi) \int_{Rw}^{c_{LH}} \left[U'(c) - U'_{LH}\right] dc \\
&= \pi \left[\int_{c_{LL}}^{c_{HH}} U'(c)dc + v_{LL}U'_{LL} - v_{HH}U'_{HH}\right] \\
&\quad - (1-\pi) \left[\int_{c_{HL}}^{c_{LH}} U'(c)dc - v_{LH}U'_{LH} + v_{HL}U'_{HL}\right].
\end{aligned}
$$

From this and Equations (21) and (22) we get

$$\delta > \pi \int_{c_{LL}}^{c_{HH}} U'(c)dc - (1-\pi) \int_{c_{HL}}^{c_{LH}} U'(c)dc = \frac{\Delta U}{2\pi - 1} > 0$$

because $\Delta U > 0$, as shown in the proof of Lemma 1, and $\pi > \frac{1}{2}$. Thus $\delta > 0$ or

$$\pi U_{HHH} + (1-\pi)U_{HHL} > \pi U_{LLL} + (1-\pi)U_{LLH}. \tag{35}$$

This and Equation (33) imply

$$\pi U_{HHH} + (1-\pi)U_{HHL} > \pi U_{LHH} + (1-\pi)U_{LHL}. \tag{36}$$

Because $\frac{S}{m_H}$ is optimal when the price is p_{HL} and because of Equation (32) it holds that

$$\pi U_{HLL} + (1-\pi)U_{HLH} > \pi U_{LLL} + (1-\pi)U_{LLH}. \tag{37}$$

Finally, from Equations (35)–(37) and $\pi > \frac{1}{2}$ it is easy to derive that $E[U(\tilde{c}_{t+1})|D_{t-1} = d_L] < E[U(\tilde{c}_{t+1})|D_{t-1} = d_H]$. Because of Assumption 4 this result contradicts the assumption $m_L \geq m_H$ and thus concludes the proof.

Proof of Proposition 7. Because of Equation (10) and Proposition 5, $m_L < m_H$ and thus $p_{LL} < p_{HL} < p_{LH} < p_{HH}$. Let $P_L(M)$ and $P_H(M)$ be the market clearing prices associated with constant market participation $M \in [\varepsilon, 1]$ and dividends d_L and d_H, respectively. Market clearing implies $P_H(M) > P_L(M)$, analogous to Equation (8). It is sufficient to show that there exists an $M' \in (m_L, m_H)$ such that $P(M') = [P_L(M'), P_H(M')]$ is less volatile than the equilibrium prices $p_{ij} = p^*(d_i, d_j)$, $i, j = L, H$. If such an M' exists, there is a neighborhood $\mathcal{I}$ of M' such that $P(M)$ is less volatile than $p_{ij} = p^*(d_i, d_j)$, $i, j = L, H$ for all $M \in \mathcal{I}$ because $P(M)$

is differentiable and thus continuous. Equation (12) implies $P_L(m_L) < p_{HL} < P_L(m_H)$; this follows from the fact that for $(M_L, M_H) = (m_L, m_H)$ the associated market clearing prices P_{ij}, $i, j = L, H$ of Lemma 2 are the equilibrium prices $p_{ij} = p^*(d_i, d_j)$. For the same reason, $P_H(m_L) < p_{LH} < P_H(m_H)$. Because $P(M)$ is continuous, there exists an $M_{HL} \in (m_L, m_H)$ and an $M_{LH} \in (m_L, m_H)$ such that

$$P_L(M_{HL}) = p_{HL}, \tag{38}$$

$$P_H(M_{LH}) = p_{LH}. \tag{39}$$

We show that $P_H(M_{HL}) < p_{HH}$ and $P_L(M_{LH}) > p_{LL}$. Because of Equation (38) and since $P_H(M_{HL})$ clears the market,

$$\begin{aligned}\pi[d_H - rP_H(M_{HL})]\, U'\left\{Rw + [d_H - rP_H(M_{HL})]\,\frac{S}{M_{HL}}\right\}\\ + (1-\pi)\,[d_H + p_{HL} - RP_H(M_{HL})]\\ \times U'\left\{Rw + [d_H + p_{HL} - RP_H(M_{HL})]\,\frac{S}{M_{HL}}\right\} = 0.\end{aligned} \tag{40}$$

Define the function $Z_H(y) := \pi(d_H - ry)\, U'[Rw + (d_H - ry)\frac{S}{M_{HL}}] + (1-\pi)(d_H + p_{HL} - Ry)\, U'[Rw + (d_H + p_{HL} - Ry)\frac{S}{M_{HL}}]$, $y \in [\frac{d_H + p_{HL}}{R}, \frac{d_H}{r}]$. Equation (40) implies

$$Z_H[P_H(M_{HL})] \quad = \quad 0. \tag{41}$$

Because of Equation (3), we have $F_{HH}(p_{LL}, p_{HL}, p_{LH}, p_{HH}, m_L, m_H) = 0$, where F_{HH} is defined in Equation (27). $Z_H(y) \equiv F_{HH}(p_{LL}, p_{HL}, p_{LH}, y, m_L, M_{HL})$, $M_{HL} < m_H$, and Equation (29) imply

$$Z_H(p_{HH}) < 0. \tag{42}$$

Differentiation of $Z_H(y)$ and applying Equation (11) gives $Z'_H(y) < 0$. This and Equations (41) and (42) imply

$$P_H(M_{HL}) < p_{HH}. \tag{43}$$

Analogously it can be shown that

$$P_L(M_{LH}) > p_{LL}. \tag{44}$$

From Equations (38) and (43) we get

$$P_H(M_{HL}) - P_L(M_{HL}) < p_{HH} - p_{HL}; \tag{45}$$

and from Equations (39) and (44) we get

$$P_H(M_{LH}) - P_L(M_{LH}) < p_{LH} - p_{LL}. \tag{46}$$

Therefore

$$P_H(M') - P_L(M') < \min(p_{LH} - p_{LL}, p_{HH} - p_{HL}) \tag{47}$$

holds either for $M' = M_{HL}$ or for $M' = M_{LH}$ or for both cases. Further, from Equations (38) and (43), $p_{LL} < p_{HL}$ and $P_H(M) > P_L(M)$ we get $p_{LL} < P_L(M_{HL}) < P_H(M_{HL}) < p_{HH}$; and because of Equations (39) and (44) and $p_{LH} < p_{HH}$, we have $p_{LL} < P_L(M_{LH}) < P_H(M_{LH}) < p_{HH}$. Thus there exists an $M' \in (m_L, m_H)$ such that Equations (13) and (14) hold for $M = M'$. As noted above, this implies Proposition 7.

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